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The Method of Complete Bifurcation Analysis for Predicting the Behavior of Complex Systems

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Abstract- **Numerous studies of nonlinear dynamical systems have revealed a wide variety of dynamic behaviors. Nonlinear effects that occur when the initial conditions and parameters change can appear in different and sometimes unexpected ways. This work focuses on studying how complex dynamics arise in nonlinear systems. A bilinear oscillator is used as an example to show how the behavior of the system changes as its parameters vary. The study is based on a systems approach and uses the method of complete bifurcation groups. To investigate nonlinear effects, analog simulation or the direct scanning method is usually employed. However, these methods are somewhat limited in their ability to predict many dynamic characteristics, including bifurcations that lead to unstable, rare, and chaotic solutions. The paper is devoted to the parametrical analysis of periodic and chaotic oscillations in non-linear dynamic systems. The description of the regular approach to construction of bifurcation diagrams is offered to your attention. This approach allows building complete bifurcation diagrams constructing into consideration regimes the analysis of which is inaccessible by traditional methods. For example, regimes with a very small area of existence are identified as rare attractors. Such approach is used in a method of complete bifurcation groups and realized in software SPRING that developing in the Riga Technical University under guidance of Professor Mikhail Zakrzhevsky. The description of this approach and results of its application are presented in this report. The object of the research is a dynamical system whose behavior is described by the second order ordinary differential equation or a system of such equations. The dynamical system state at any time moment is characterized by the values of its phase coordinates and behavior – by variation of the phase coordinates in time. The behavior of a determinate dynamical system is uniquely defined by its initial state and the fixed parameters. The initial state is given by the values of the phase coordinates at the start. In the system behavior we can distinguish a transient process, which ends by a stationary state or regime. The stationary regime can be periodic with a period equal to the mapping period, subharmonic with a period multiple to the mapping period, almost periodic and chaotic. This method is effective for scientific research because it allows a systematic study of the main scenarios of birth and conditions for the formation of chaotic and rare attractors, which depend on parameters defined by engineering systems. Applications of this method include the simulation and analysis of nonlinear dynamic systems, the prediction of technical disasters, and the development of new vibration technologies and other technical systems.**

Keywords- complete bifurcation diagram, rare attractor, chaos.

I. INTRODUCTION

The study of nonlinear dynamical systems is an important task in science and engineering, as their behavior can change in complex and sometimes unexpected ways when parameters or initial conditions vary [1] – [2]. Traditional analysis methods, such as analog simulation and direct scanning, have limited capabilities in predicting rare and chaotic regimes [3] – [6]. Therefore, the development and application of the method of complete bifurcation groups [7] – [9] is a relevant scientific challenge, allowing for a more detailed investigation of the mechanisms behind the emergence of complex dynamics.

The purpose of this study is to investigate the processes of complex dynamical regime formation in nonlinear systems using a bilinear oscillator as an example. This research relies on the method of complete bifurcation groups [10] – [12], which allows for the construction of bifurcation diagrams containing both stable and unstable periodic regimes, as well as the systematic identification of dynamic regimes inaccessible to traditional analysis methods. These diagrams will be referred to as "complete bifurcation diagrams". The obtained results can be applied in the simulation and analysis of nonlinear systems to predict their behavior, prevent technical disasters, and develop new vibration technologies and other engineering systems.

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The object of the research is a dynamical system whose behavior is described by the second order ordinary differential equation or a system of such equations. The dynamical system state at any time moment is characterized by the values of its phase coordinates and behavior – by variation of the phase coordinates in time (time history). The behavior of a determinate dynamical system is uniquely determined by the initial state (the value of the phase coordinates at the initial time) and by the values of fixed parameters. Traditionally a system behavior is represented in the form of a continuous phase trajectory, but for some researches only points of discrete mapping of this trajectory can be used, as suggested by French mathematician Poincaré. In this case only certain points can be retained from continuous phase trajectory, located at equal time intervals, which can be called a period of mapping. In the non-autonomous systems with periodic excitation the mapping period is chosen to be equal to the period of driving force. This method of mapping is called Poincaré mapping or T-mapping.

In the system behavior we can distinguish a transient process, which ends by a stationary state or regime. The stationary regime can be periodic with a period equal to the mapping period (P1), subharmonic with a period multiple to the mapping period (P2, P3…), almost periodic (PA) and chaotic (Cha). In the case of periodic and subharmonic state the discrete mapping points are repeated, forming fixed points in Poincaré map. In the case of non-periodic steady state, the discrete mapping points fill a fixed region of phase space, not going beyond its borders.

The general characteristics of a steady state do not change over time. For example, phase coordinates of fixed points, phase portraits, amplitude of oscillations, amplitude spectrum, etc. A set of such characteristics we call a passport of steady state.

Characteristics in passport of steady state are unchanged in time, but will change with the variation of system parameters. For example, the amplitude of oscillations may be dependent on the frequency of the driving force or the stiffness of the elastic coupling. The study of such dependencies is the purpose of a parametric or bifurcation analysis. Typically, the bifurcation analysis results are presented in the form of bifurcation diagrams, which show the change of the characteristics of the stationary state when changing one or more system parameters. For example, the frequency response is the bifurcation diagram showing a dependency of the stationary oscillations amplitude on the forcing frequency.

Thus, the purpose of this paper is to present the method of bifurcation analysis and to describe the typical topology of bifurcation diagrams of forced dynamical systems.

This method is applied to investigate fundamental scenarios of attractor formation and the conditions for the emergence of chaotic and rare attractors, depending on parameters determined by engineering systems. The practical applications of this method include the analysis and simulation of nonlinear dynamical systems, the prediction of technical failures, the development of novel vibration-based technologies, and various other engineering applications.

II. Materials and Methods

In this study, a bilinear oscillator was analyzed to investigate the formation of complex dynamic regimes in nonlinear systems. The method of complete bifurcation groups was employed to construct comprehensive bifurcation diagrams, facilitating the systematic identification of dynamic regimes that are often undetectable by traditional analysis methods. This approach enables a deeper understanding of the mechanisms underlying complex dynamics in nonlinear systems.

A. Researched model

The results of bifurcation analysis of forced oscillations in the dynamical system with bilinear elastic force are considered in this paper. The scheme of this model is shown in Fig. 1.

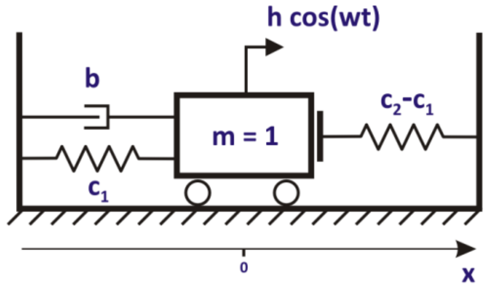

Fig. 1. Forced dynamical system with bilinear elastic force and viscous friction.

The behavior of the researched dynamical system is described by the second order differential equation (1).

$$m\ddot{x} + F_v(\dot{x}) + F_x(x) = F_t(t) \quad (1)$$

The dynamical system contains three forces (Fig.2): bilinear restitution force $F_x(x)$, harmonical excitation force $F_t(t)$ and linear dissipative force $F_v(\dot{x})$.

$$F_x(x) = \begin{cases} c_1 x, & x < 0 \\ c_2 x, & x \geq 0 \end{cases} \quad (2)$$

$$F_t(t) = h\cos(wt) \quad (3)$$

$$F_v(\dot{x}) = b\dot{x} \quad (4)$$

The parameters of the dynamical system are: m = 1 – body mass, b = 0.05 – coefficient of linear dissipative force, $c_1 = 1$ and $c_2 = 16$ – stiffness of elastic couples, h = 1 and w = 0.5…8.5 – amplitude and frequency of external harmonical excitation.

The detailed description of the forced dynamical system—with its bilinear elastic force, viscous friction and harmonical excitation — along with the robust numerical methods employed to solve its governing equation, provides a solid foundation for analyzing its dynamic behavior.

With the model and experimental procedures clearly defined, the discussion shifts to the intricate patterns that emerge as system parameters vary.

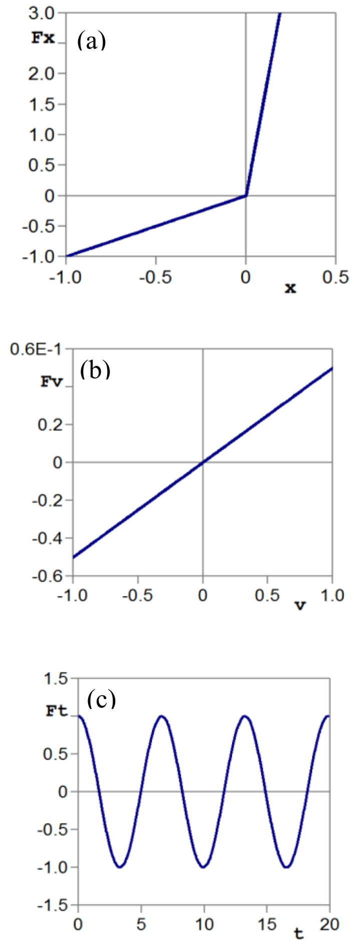

Fig. 2. The graphs of researched model's forces: (a) bilinear elastic force Fx(x), (b) linear dissipative force Fv(x΄) and (c) harmonical excitation force Ft(t).

The next section delves into the construction of complete bifurcation diagrams, which capture the evolution of periodic and chaotic regimes, offering deeper insights into the underlying mechanisms driving these complex dynamics.

B. *Methods of research*

The given research is based on the numerical solution of the Cauchy problem for the differential equation (1) from given initial condition at fixed parameters values. The resulting time history, obtained using the numerical integration by the Runge-Kutta method, is separated into transitional process and stationary state. For example, Fig. 3 shows time history that can be realized in the considered system at excitation frequency w = 0.88. Time history, phase portrait and Poincaré mapping illustrate the transitional process that ended by periodic steady state with the period of excitation force. Such periodic attractors we call P1. On the phase plane this periodic regime corresponds to the limit circle, on the Poincaré map – to the fixed point.

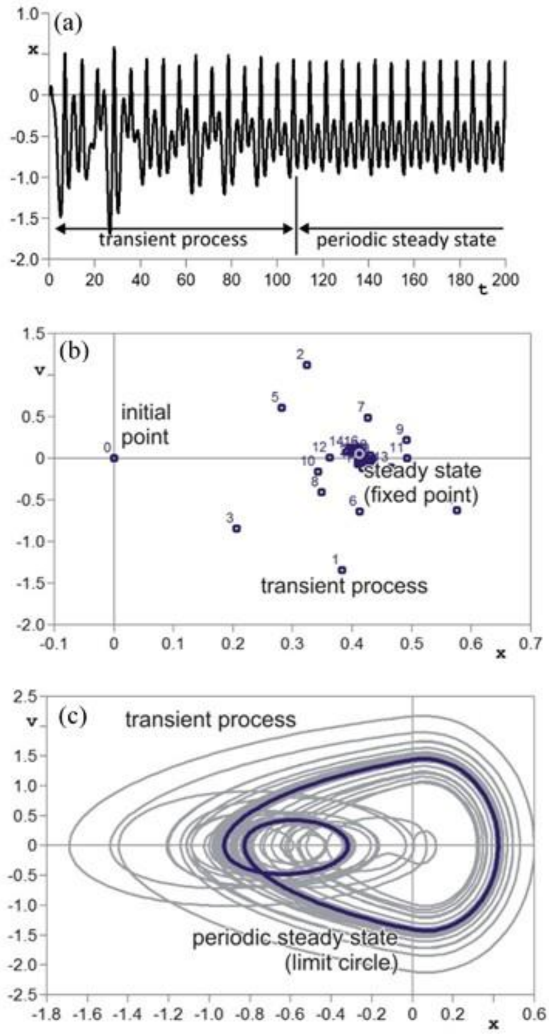

Fig. 3. Time history (a), phase portrait (b) and discrete mapping points (c) of transient process and stationary periodic regime P1. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.88. Initial conditions x_0=0, v_0=0.

It is more convenient to solve the problem of periodic regimes searching for the discrete equation obtained from the original differential equation on the basis of the Poincaré map. For this, it is necessary to substitute a continuous system described by the differential equation (1) with a discrete system described by the discrete mapping operator D, which defines the change in the state of the system over one period of the investigated regime (5) [13], [14].

$$X(t+T) = D(X(t)), \quad (5)$$

where X(t) is the system time history, T is the mapping period, for the considered non-autonomous system it's equal to the period of the driving force $T = \frac{2\pi}{w}$, D is discrete mapping operator, describing the variations of the system state via the mapping period.

For most systems obtaining of the discrete mapping operator analytically is impossible. In this research for the considered system the discrete mapping operator is constructed algorithmically based on the numerical integration of the original differential equation (1) over the mapping period T.

To find the periodic solution P1 of the original differential equation (1) it is necessary to solve the discrete equation (6).

$$X = D(X) \qquad (6)$$

Roots of the equation (6) will be fixed points of the discrete mapping or initial conditions of periodic regimes P1. To find periodic solutions with longer periods (P2, P3…) it is enough to apply the discrete mapping operator in equation (6) appropriate number of times. For example, for regimes P2 – two times (7).

$$X = D(D\,X) \qquad (7)$$

For the considered system equations (6) and (7) can be solved only numerically, e.g., using Newton's method.

The roots of the equations (6) and (7) correspond to the initial conditions of periodic regimes. We call the set of basic characteristics of such regimes the regime's passport. For example, the passport of the periodic regime may include: order, amplitude and period, phase portrait and limit values of phase coordinates, coordinates of fixed points, characteristics of stability, results of spectral analysis, etc. Passport of stable first order periodic regime P1 that exists in considered system at excitation frequency w = 0.88 is shown in Fig. 4.

The obtained solutions correspond to both stable and unstable periodic regimes. Evaluation of the periodic regime's stability is based on the Floquet theory [15], according to which the periodic solution is stable if all roots λ of the characteristic equation of monodromy matrix (Floquet multipliers) (8), calculated for the discrete mapping operator (5) at a fixed point, will be less than one in absolute value.

$$det\left[\frac{dD(X)}{dX} - \lambda E\right] = 0 \qquad (8)$$

For example, the regime in Fig. 4 is stable, since the absolute values of both multipliers are less than one. This is easily seen on the argand diagram, where the unit circle and the complex roots of the monodromy matrix are mapped.

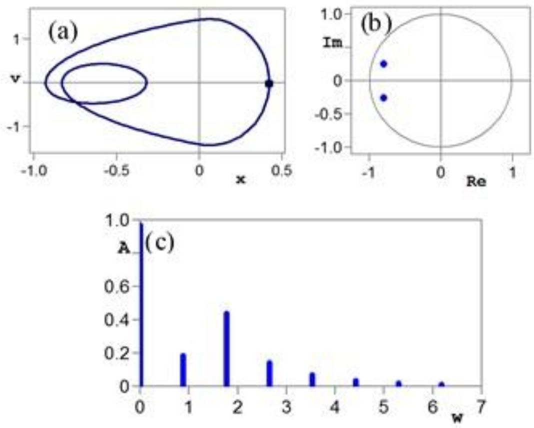

Fig. 4. Passport of the periodic regime P1. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.88.
(a) Initial conditions: t0 = 0; x0 = 0.421642; v0 = -0.029472.
(b) Floquet multipliers: λ1 = -0.79736 + 0.252962i;
λ2 = -0.79736 - 0.252962i
(c) Spectrum: main harmonics (A > 0.1)

To perform parametric analysis and to construct bifurcation diagrams we used the method of complete bifurcation groups [7]-[9], which is based on the method of parameter continuation for the stable and unstable periodic solutions.

The graphs, describing changes of the system steady state when one of its parameters is changed, are called one-parameter bifurcation diagrams. The diagram may present changes of the fixed point's phase coordinates ($x, \dot{x} = v$) or the periodic regime's amplitude (Ax) when changing a system parameter. Practically the construction of such diagrams means the solution of equations (6) and (7) for different values of the variable parameter. This method is known as the method of parameter continuation. Its application allows one to represent points corresponding to both stable and unstable periodic regimes on the complete bifurcation diagrams [12].

For example, the bifurcation diagram for regime P1, whose passport is shown in Fig. 4, is shown in Fig. 5. This diagram shows the change of the regime amplitude when the frequency of the driving force varies from 0.8 to 1.9. In the diagram a resonance peak, which for the system under consideration is at the frequency w = 1.6, is clearly allocated. The diagram shows the amplitudes of both the stable and unstable regimes for which the Floquet multipliers are greater than one. The points, at which the stability of the regime changes, are called bifurcation points. At these points in addition to changing the stability other oscillation regimes are born, for which the own branches of the bifurcation diagram can be constructed. The bifurcation diagram, which contains branches for all stable and unstable oscillation regimes realized in the system, can be called a complete bifurcation diagram.

On the complete bifurcation diagram particular importance is given to the bifurcation points in which a qualitative change of the regime's stability occurs. For example, the period-doubling bifurcation at stability loss of regime P1 leads to the birth of stable regime P2.

Sequence of period-doubling bifurcations is one of the possible scenarios for the birth of chaos [7], [16], [17], known as the Feigenbaum scenario [18]. The cascade of period-doubling bifurcations leads to appearance of an infinite number of unstable periodic solutions (e.g. P1, P2, P4, P8...). Such structure will be called as unstable periodic infinitium (UPI). Presence of UPI in the system determines the existence of a stationary or transient chaos [7].

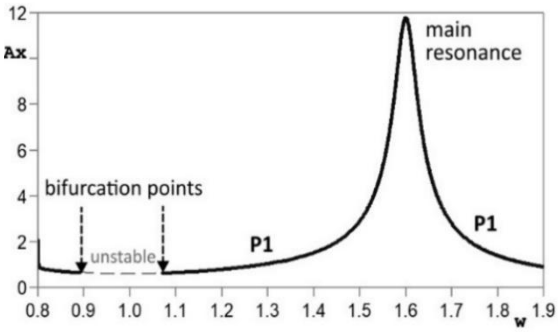

Fig. 5. Bifurcation diagram of periodic regime P1 shows the changes of regime amplitude Ax at variation of excitation frequency w. Diagram contains fragments, corresponding to stable and unstable solutions. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.88…1.9.

On the plane of the two parameters a two-parameter bifurcation diagram can be constructed. This diagram consists of bifurcation boundaries, every point of which corresponds to the bifurcation point on the one-parameter diagram. Bifurcation boundaries corresponding to the bifurcation of period-doubling cascade limit in the two-parameter diagram regions of parameters at which UPI exists. In these regions chaotic behaviour of the system is possible. To construct a two-parameter bifurcation diagrams a method of movement on the bifurcation boundary can be used.

The following section delves into the construction of complete bifurcation diagrams, which capture the evolution of periodic and chaotic regimes, offering deeper insights into the underlying mechanisms driving these complex dynamics.

C. Complete bifurcation diagram

The construction of a complete bifurcation diagram begins with the search of all possible periodic regimes at fixed parameters values. It is important to find not only stable, but also unstable periodic solutions. For this discrete equations similar to (6) and (7) need to be solved. As initial conditions for the iterative processes of numerical solutions, points from a certain region of the phase plane are chosen. As we shall see hereinafter, it is important to find at least one regime from each bifurcation group. The remaining regimes will be found when constructing the complete bifurcation diagram by the method of parameter continuation.

Passports of all found regimes form a periodic skeleton. It is the basis for constructing a complete bifurcation diagram. An example of a periodic skeleton for the system under consideration is shown in Fig. 6. It contains three periodic regimes, two of which are unstable.

Applying the method of parameter continuation and the hypothesis about the continuity of the bifurcation diagrams branch, it is possible to construct a branch of the bifurcation diagram for each regime from the periodic skeleton. In the bifurcation points of the branch, new periodic regimes can be born, which will be the starting points for constructing new branches. The set of branches of the bifurcation diagram obtained in this way is called the bifurcation group. The order of the bifurcation group is determined by the regime with the lowest order. The bifurcation branch of this regime is either infinite in the direction of decreasing and increasing the parameter (this is characteristic of the basic regime P1), or it is closed to itself. Closed groups are called island-type bifurcation groups or islands.

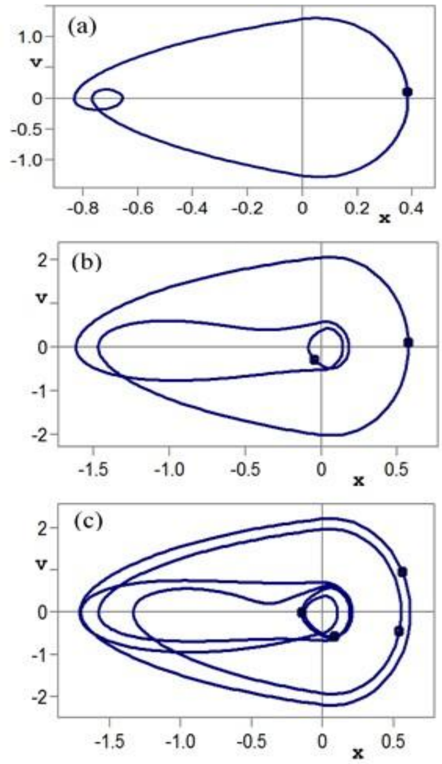

Fig. 6. **P**eriodic skeleton – the passports of all possible periodic regimes (P1,P2 and P4) at given fixed parameters values:
(a) unstable periodic regime P1;
(b) unstable periodic regime P2;
(c) stable periodic regime P4.
For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=1.01.

An example of a complete bifurcation diagram for the system under consideration is shown in Fig. 7. The diagram contains one bifurcation group of order P1. It includes stable and unstable branches for periodic solutions P1, P2, P4 and P8. Moreover, solutions of higher orders are produced at bifurcation points of lower-order solutions, when the latter lose their stability. The sequence of such period doubling bifurcations (of which only the first three are shown in the diagram) leads to the creation of an infinite number of unstable solutions branches. We call this structure UPI (Unstable Periodic Infinitium). It is the cause of the chaotic behavior of the system, which can be expressed either in a chaotic attractor or in a chaotic transient process. An example of the discrete mapping of the chaotic stationary regime from the UPI region of the bifurcation diagram of Fig. 7 is shown in Fig. 8.

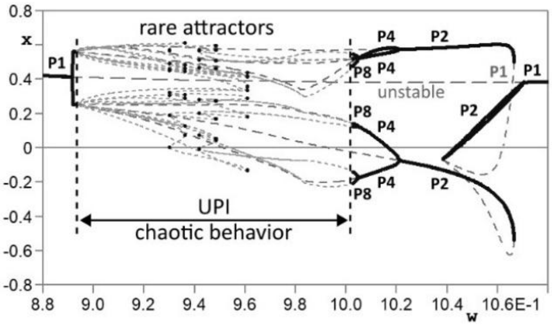

Fig. 7. Complete bifurcation diagram of bifurcation group P1 shows the changes of the fixed points coordinates x at variation of excitation frequency w in low frequency region. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.88…1.08.

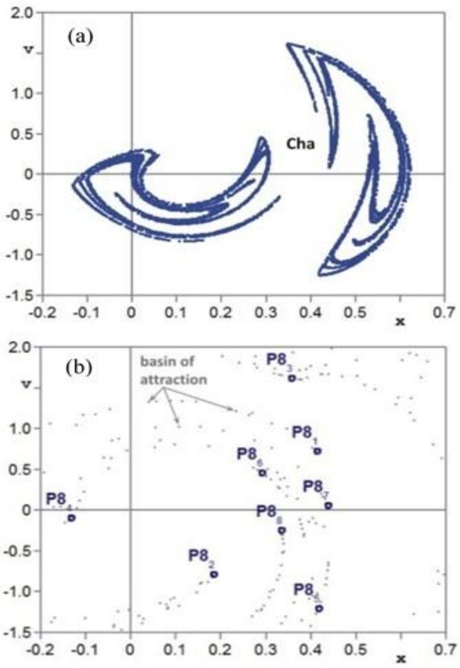

Fig. 8. Poincaré mapping of chaotic attractor (a) contains 10 000 points from initial condition $x_0 = 0$, $v_0 = 0$ and fixed points (b) of coexisted periodic rare attractor P8 (Fig. 9) with very narrow basin of attraction. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.9608.

The bifurcation diagram in Fig. 7 illustrates one more phenomenon, which is revealed with the help of complete bifurcation diagrams – these are rare attractors. On the unstable branch of the P8 regime of this diagram small regions of stability are seen. In these areas, the P8 regime is stable in very narrow parameter ranges. We call such narrow regions of stability as rare attractors. Fig.7 shows five rare attractors that are stable at the values of the parameter w showing in Table 1.

TABLE 1 PARAMETER VALUES FOR RARE ATTRACTORS P8

Rare attractor	Minimal w value	Maximal w value
RA_1	0.93013	0.93014
RA_2	0.93617	0.93618
RA_3	0.94175	0.94176
RA_4	0.94839	0.94840
RA_5	0.96079	0.96082

The passport of one of these rare attractors (at w = 0.96080) is shown in Fig. 9. Due to the fact that this rare attractor is in the UPI region, it coexists with a chaotic regime, the discrete mapping of which is shown in Fig. 8. A rare attractor has a very small region of attraction, which also complicates its detection. Practically it can be found only with the help of a complete bifurcation diagram.

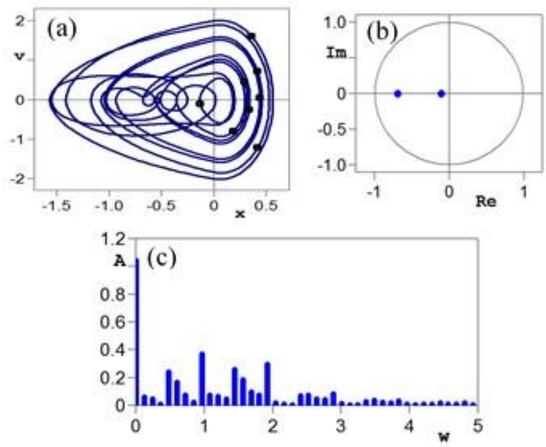

Fig. 9. Passport of the rare attractor P8 that is stable at very narrow diapason of excitation frequency w=0.96079…0.96082. For bilinear system (1) at parameters m=1, c1=1, c2=16, b=0.05, h=1, w=0.96080.
(a) Initial conditions: t0 = 0; x0 = 0.413051; v0 = 0.728654.
(b) Floquet multipliers: $\lambda 1 = -0.107133 + 0i$;
$\lambda 2 = -0.687102 + 0i$.
(c) Spectrum: main harmonics (A > 0.2)

To construct bifurcation diagrams of other bifurcation groups it is necessary to construct periodic skeletons for other values of the parameter and, starting from the periodic regimes of these skeletons, construct branches of other bifurcation groups. Fig. 10 shows a complete bifurcation diagram containing all bifurcation groups of orders from 1 to 5 over a wide range of frequencies of the driving force.

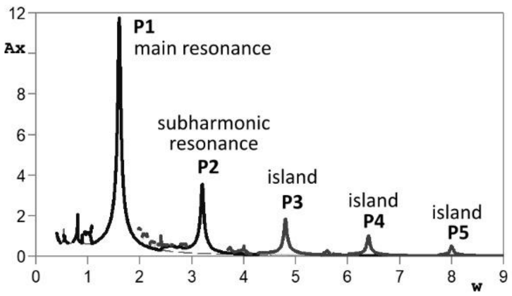

Fig. 10. Complete bifurcation diagram shows the changes of the periodic regimes amplitudes Ax at changing of excitation frequency w. For bilinear system (1) at parameters m=1, c_1=1, c_2=16, b=0.05, h=1, w=0.5…8.5.

Thus, the construction of a complete bifurcation diagram provides a comprehensive representation of the system's dynamic behavior, capturing both stable and unstable periodic solutions. The identification of bifurcation groups and their hierarchical structure allows for a deeper understanding of the transitions between periodic and chaotic regimes. Moreover, the discovery of rare attractors highlights the necessity of this approach, as such regimes are difficult to detect with conventional methods. The complete bifurcation analysis not only enhances our theoretical understanding of nonlinear dynamics but also has significant practical implications for the control and prediction of complex systems.

III. Results and discussion

The research presented in this paper demonstrates the effectiveness of the complete bifurcation analysis method in predicting the behavior of nonlinear dynamical systems. By constructing complete bifurcation diagrams, including stable and unstable periodic solutions, we gain deeper insights into the system’s behavior beyond traditional bifurcation analysis. The method reveals fundamental mechanisms of attractor formation, chaotic transitions, and rare attractor existence, significantly improving the accuracy of dynamical system simulation.

The bifurcation diagrams constructed for a bilinear elastic force system illustrate the transition from periodic oscillations to chaotic behavior via the period-doubling cascade, confirming the Feigenbaum scenario. Additionally, the study highlights the concept of Unstable Periodic Infinitium (UPI), a structural feature of bifurcation diagrams that plays a crucial role in the emergence of chaos. The identification of rare attractors with extremely narrow stability regions further underscores the necessity of the complete bifurcation approach, as these attractors are difficult to detect using conventional methods. The existence of these rare attractors demonstrates the complex and sensitive dependence of dynamical systems on parameter variations.

Moreover, the results obtained using numerical continuation techniques emphasize the importance of parameter continuation for capturing both stable and unstable solutions. The approach facilitates the identification of bifurcation points, where qualitative changes in stability occur, leading to the birth of new periodic regimes. This capability is critical for engineering applications, including the prediction of technical failures, vibration control, and the development of nonlinear mechanical systems.

IV. Conclusions

This study employs the method of complete bifurcation groups to systematically analyze the complex dynamics of nonlinear systems. A bilinear oscillator serves as the primary model for investigation. The research utilizes the SPRING software, developed at Riga Technical University under the guidance of Professor Mikhail Zakrzhevsky, to construct comprehensive bifurcation diagrams. This approach enables the identification of dynamic regimes, including rare attractors, that are often undetectable using traditional analysis methods. The methodology facilitates a deeper understanding of the mechanisms leading to complex behaviors in nonlinear dynamical systems.

This research represents the method of complete bifurcation analysis as a powerful tool for investigating the dynamics of nonlinear systems. The method allows for the systematic exploration of stable and unstable periodic solutions, revealing complex bifurcation structures such as period-doubling cascades, chaotic attractors, and rare attractors. The findings confirm that Unstable Periodic Infinitium (UPI) plays a crucial role in the emergence of chaotic behavior and highlight the importance of capturing rare attractors, which can significantly influence system stability.

The proposed methodology is not only theoretical but also has practical implications in various engineering domains. It enables more accurate simulation of nonlinear oscillatory systems, assists in predicting critical transitions leading to instability, and provides valuable insights for the design and optimization of engineering structures. Future research can extend this method to explore higher-dimensional dynamical systems, further enhancing our understanding of complex nonlinear phenomena.

The presented study highlights the effectiveness of the method of complete bifurcation groups in predicting the behavior of nonlinear dynamic systems. Constructing complete bifurcation diagrams that include both stable and unstable periodic solutions provides a deeper understanding of the system's dynamics, extending beyond traditional analyses.

The method of complete bifurcation groups enables the identification of fundamental mechanisms behind attractor formation, transitions to chaos, and the existence of rare attractors, thereby deepening our theoretical understanding of dynamic systems. Analyzing interactions between bifurcations across various periodic regimes enhances our comprehension of universal patterns in the dynamics of nonlinear systems. Furthermore, the consistent detection of attractors with extremely narrow stability regions highlights the complex and sensitive dependence of dynamic systems on parameter variations, which is crucial for chaos theory.

Practically, this method aids in predicting technical failures; understanding bifurcation mechanisms and chaotic behavior allows for anticipating potential failures in engineering systems, thereby increasing their reliability. Studying dynamic transitions also contributes to developing effective vibration control methods in mechanical systems, which is vital for ensuring stability and safety. Lastly, a comprehensive understanding of nonlinear dynamic behaviors facilitates the design of more efficient and robust mechanisms in various engineering applications.

In summary, the research underscores the importance of the method of complete bifurcation groups as both a theoretical tool for gaining profound insights into the dynamics of complex systems and a practical instrument for developing and optimizing engineering solutions.

V. ACKNOWLEDGMENTS

The authors express their sincere gratitude to Professor Mikhail Zakrzhevsky for fostering a creative and inspiring research environment. His guidance and support have been instrumental in the development of the method of complete bifurcation groups, enabling the systematic study and analysis of complex dynamics in nonlinear systems. Additionally, his visionary leadership made it possible to implement the ideas of this method in the software SPRING, further advancing the research in this field.

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