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Modeling and Numerical Solution of Liquid Filtration Processes in Multilayer Oil Fields

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***Abstract*-The paper presents the physic and mathematic models for fluid filtration in the multilayered strata with poorly permeable and well permeable layer. Developed computational algorithm and program for the intermediate model. Shows computer experiment compared the results with other modules for the development of a multilayered reservoir field. Based on analysis numerical results is that the approval of the interim model is optimal in terms of efficiency of computer time and memory consumption of computer technology.**

Keywords-mathematic models, fluid filtration, reservoir field.

I. INTRODUCTION

Many oil and gas condensate fields are a set of deposits, i.e. they are multilayered. In some cases, it is advisable to subdivide the productive horizon into separate units, especially if they are separated from each other by consistent interlayers, such as clay.

For the theory and practice of developing multilayer fluids in fields, it is important to establish the permeability, low permeability or impermeability of the separating bridges. In some cases, this question can be answered using exploratory well data.

If the following conditions are met simultaneously:
1) the distribution of initial reservoir pressures by horizons obeys the barometric formula:
2) gas-water contacts are at the same elevation:
3) the compositions of the fluids in all horizons are the same, then we can reliably say that a hydrodynamic connection has manifested itself at least over geological time. In such cases, when carrying out forecast calculations, one should keep in mind the possibility of a hydrodynamic connection manifesting itself during the development of a multi-layer field.

If the specified conditions are not met, and if the initial pressure in the horizons is distributed according to the hydrostatic law, we can confidently speak of the isolation of the productive horizons from each other. With the specified identification, we should keep in mind the possibility of the absence of a hydrodynamic connection between the layers and the presence of a hydrodynamic connection in the area of water content. Then one of the listed conditions can be met.

Many studies have been conducted on modeling the processes of oil extraction from multi-layer oil fields. The mathematical modeling of fluid filtration in multicomponent fluids including water, gas, and oil as well as in heterogeneous multilayer porous media is the main topic of this research article. It explains the difficulties in resolving multidimensional, nonlinear partial differential equations and looks at several simplifications for the creation of useful models. The scientific work considers existing models of systems incorporating faults, two- and three-phase flows, nanoporous rocks, and man-made substances. Unsteady filtering processes in two-layer systems, whose dynamics rely on permeability, diffusion, and layer thickness, are given special consideration. The study provides methods for improving oil and gas extraction procedures and advances knowledge of the condition of liquids in porous media [1,2,3].

In this study, a mathematical model for examining liquid filtration in horizontal wells in a sealed type of

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formation is developed. The finite element technique (FEM) was used to examine the effects of permeability, deformation, and anisotropy on well performance. Numerical models demonstrate that variables like vertical permeability and isotropy angle have a major impact on flow rate. The software tools and algorithms that have been developed offer useful applications for improving drilling tactics and raising reservoir productivity. A 30.8% increase in productivity during field application demonstrated the model's efficacy in boosting oil and gas extraction [4,5].

Let there be a three-layer interacting layer with a region D, the middle one is highly permeable, and two extreme ones are poorly permeable (Fig. 1). It is required to determine a continuous function of pressure $P_i(x, y, z, t)$ satisfying a system of differential equations

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}\left(K_1(x,y,z)\frac{\partial P_1}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_1(x,y,z)\frac{\partial P_1}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_1(x,y,z)\frac{\partial P_1}{\partial z}\right)=M_1(x,y,z)\frac{\partial P_1}{\partial t}\\
&\frac{\partial}{\partial x}\left(K_2(x,y,z)\frac{\partial P_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_2(x,y,z)\frac{\partial P_2}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_2(x,y,z)\frac{\partial P_2}{\partial z}\right)=\\
&=M_2(x,y,z)\frac{\partial P_2}{\partial t}+F(x,y,z,t)\\
&\frac{\partial}{\partial x}\left(K_3(x,y,z)\frac{\partial P_3}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_3(x,y,z)\frac{\partial P_3}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_3(x,y,z)\frac{\partial P_3}{\partial z}\right)=M_3(x,y,z)\frac{\partial P_3}{\partial t}
\end{aligned}\right\} \quad (1)$$

under the initial

$$P_i(x, y, z, 0) = \varphi_i(x, y, z) \qquad (2)$$

and boundary conditions

$$\left.\frac{\partial P_i}{\partial n}\right|_{\Gamma} = 0 \qquad (3)$$

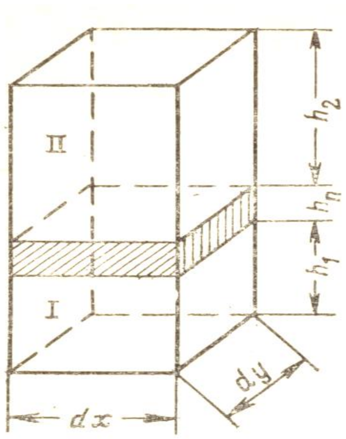

Fig. 1. Elementary volume in the presence of a well-permeable bridge between the layers

Here P_i - is the pressure, K_i, M_i, F_i -are the given functions of their arguments, $i = 1,2,3$, n -is the external normal to G, G- is the surface of the filtration region under consideration D [6,7]. As a function $\varphi_i(x, y, z)$, a constant value is often taken that characterizes the undisturbed state of the formation at the initial moment of time, neglecting the weight of the liquid column within the thickness of the i-th interlayer. According to the Hantusch model, instead of solving problem (1)-(3), we have a more simplified problem

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}\left(K_2\frac{\partial \vartheta_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_2\frac{\partial \vartheta_2}{\partial y}\right)=M_2\frac{\partial \vartheta_2}{\partial t}+F(x,y,t)-q_2;\\
&\vartheta_2\big|_{t=0}=\varphi_2(x,y);\;\; P_i\big|_{t=0}=\varphi_i(x,y,z);\;\; P_1\big|_{z=h_1-0}=\vartheta_2\big|_{z=h_1+0};\\
&P_3\big|_{z=h_2+0}=\vartheta_2\big|_{z=h_2+0};\;\; q_2=q_1+q_3;\;\; q_i=\frac{K}{h_i-h_{i-1}}\left.\frac{\partial P_i}{\partial z}\right|_{z=h_i};\\
&\vartheta_2=\frac{1}{h_2-h_1}\int P_2(x,y,z,t)dz.
\end{aligned}\right\} \quad (4)$$

where q_i is the flow, $i = 1,3$.

Thus, instead of the spatial initial problem (1)-(3), a comparatively simple quasi-three-dimensional problem (4) was obtained. In the work [5-8], the problem in this formulation is considered, the methods of solution in test data are given, the results are obtained and analyzed. However, researchers still face difficulties in solving the problem according to the obtained model, taking up more memory and time. For this purpose, we use a somewhat simplified model than the Hantush model, an intermediate model [1, 8].

In the task (4), averaging the movements by spatial variables to determine the values of the flow between layers and the change in the pressure value in the bridges, we obtain the so-called material balance model

$$\frac{\partial}{\partial z}\left[K_i(z)\frac{\partial \vartheta_i}{\partial z}\right]=M_i\frac{\partial \vartheta_i}{\partial t};\;\; h_{i-1}<z<h_i;\;\; t>0. \quad (5)$$

$$\vartheta_i\big|_{t=0}=\varphi_i(z);\;\; z\in[h_{i-1},h_i]. \quad (6)$$

$$\left.\frac{\partial \vartheta_i}{\partial z}\right|_{z=h_0,h_3}=0;\;\; \vartheta_i\big|_{z=h_1,h_2}=\overline{\vartheta_2}. \quad (7)$$

$$\vartheta_i=\int_0^1\int_0^1 P_i(x,y,z,t)dxdy;\;\; i=1,3.$$

here, $\overline{\vartheta_2}=\sum_{l=1}^{N_x}\sum_{j=1}^{N_y}\frac{\vartheta_{2l,j}}{N_x\cdot N_y}$, where N_x, N_y is the number of nodes along X, Y.

The change in the pressure field in a well-permeable formation is determined by solving the equation

$$\frac{\partial}{\partial x}\left(K_2\frac{\partial \vartheta_2}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_2\frac{\partial \vartheta_2}{\partial y}\right)=M_i\frac{\partial \vartheta_2}{\partial t}+F(x,y,t)-q_{n2} \quad (8)$$

with the corresponding initial and boundary conditions.

II. MATERIALS AND METHODS

To solve the problem numerically, we move on to dimensionless variables:

$$\overline{K}=\frac{K}{K_x},\quad \overline{\mu}=\frac{\mu}{\mu_x},\quad \overline{x}=\frac{x}{L_x},\quad \overline{y}=\frac{y}{L_y},\quad \overline{z}=\frac{z}{L_z}\quad \overline{P}=\frac{P}{P_x},$$

$$Q=A\sum_{i=1}^{N}\delta(x-x_i,y-y_i)q_i,$$

$$A=\frac{L_x\mu_x}{K_xP_xL_yL_z},\quad t=\tau\frac{K_x}{\mu_xL_x^2},\quad Qp_x=\frac{L_x\mu_x}{K_xP_xL_yL_z}q_{n2}. \quad (9)$$

For ease of notation, we omit the dashes above the variables.

Here is the q_{n2}-function of the flow entering as a uniformly distributed internal drain [9].

Thus, the complicated original problem (4) is reduced to a sequential solution of the one-dimensional (5) and two-dimensional problem (8).

To solve the problem (8), we use the method of the longitudinal-transverse scheme [10,11] and the flow version of the sweep method.

For this, we cover the given two-dimensional region with a uniform grid:

$$\omega_{x,y}=\{(x_i=i\cdot h_x,\ h_x=\frac{1}{N_x}\ y_j=j\cdot h_y,\ h_y=\frac{1}{N_y}\ ),\ i=\overline{1,N_x},\ j=\overline{1,N_y},\ \}$$

In equation (8) we introduce variables in the form for applying the flow sweep:

$$w_x=K\frac{\partial \vartheta_2}{\partial x},\quad w_y=K\frac{\partial \vartheta_2}{\partial y}.$$

Then (8) takes the following form:

$$\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}=m\frac{\partial \vartheta_2}{\partial t}+Q-q_{n2} \quad (10)$$

The finite difference form of equation (10) when calculated using the longitudinal-transverse scheme is as follows:

$$\left(w_{x_{i+\frac12,j}}^{k+\frac12}-w_{x_{i-\frac12,j}}^{k+\frac12}\right)=m_{i,j}\frac{h_x}{\tau}\vartheta_{2i,j}^{k+\frac12}+\frac{h_x}{\tau}F_i;\quad F_i=\tau\cdot\left(Q_{i,j}-\left(q_{n2}+\frac{w_{y_{i,j+\frac12}}^{k}-w_{y_{i,j-\frac12}}^{k}}{h_y}\right)\right)-m_{i,j}\vartheta_{2i,j}^{k};$$

$$\left(w_{y_{i,j+\frac12}}^{k+1}-w_{y_{i,j-\frac12}}^{k+1}\right)=m_{i,j}\frac{h_y}{\tau}\vartheta_{2i,j}^{k+1}+\frac{h_y}{\tau}F_j;\quad F_j=\tau\cdot\left(Q_{i,j}-\left(q_{n2}+\frac{w_{x_{i+\frac12,j}}^{k+\frac12}-w_{x_{i-\frac12,j}}^{k+\frac12}}{h_x}\right)\right)-m_{i,j}\vartheta_{2i,j}^{k+\frac12}.$$

(11)

In equation (5) we introduce variables in the form for applying the flow sweep [12,13]:

$$w_z=K\frac{\partial \vartheta_i}{\partial z},\quad i=1,3.$$

Then (5) takes the following form:

$$\frac{\partial w_z}{\partial z}=m\frac{\partial \vartheta_i}{\partial t},\quad i=1,3 \quad (12)$$

Finite difference form of equation (12):

$$\left(w_{z_{j+\frac12}}^{k+1}-w_{z_{j-\frac12}}^{k+1}\right)=m_j\frac{h_z}{\tau}\vartheta_{ij}^{k+1}+\frac{h_z}{\tau}F_i,\quad F_i=-m_j\vartheta_{ij}^{k},\quad i=1,3. \quad (13)$$

In parameterized form, the boundary conditions can be written as:

$$\left[\lambda\frac{\partial \vartheta_i}{\partial n}+(1-\lambda)\vartheta_i\right]_{\Gamma}=\gamma,\quad i=1,2,3.$$

The parameter λ takes the value 0 or 1 (for $\lambda=0$ the first boundary condition, for $\lambda=1$ the second boundary condition).

To calculate (11) and (13) using the flow sweep method [13-14], we use the following algorithm:

$$\left.\begin{array}{l}
\alpha_N=\dfrac{-\lambda_2}{0.5\cdot\lambda_2+(1-\lambda_2)},\quad \beta_N=\dfrac{\lambda_2-0.5\cdot\lambda_2*F_N}{0.5\cdot\lambda_2+(1-\lambda_2)},\\
j=(N_x-1)...0,\quad j=(N_y-1)...0,\\
\alpha_j=\dfrac{\frac{h^2}{\tau}-\alpha_{j+1}}{1+\frac{h^2}{\tau}-\alpha_{j+1}},\quad \beta_j=\dfrac{\beta_{j+1}-F_j\left(\frac{h^2}{\tau}-\alpha_{j+1}\right)}{1+\frac{h^2}{\tau}-\alpha_{j+1}},\\
\vartheta_{i0}=\dfrac{\frac{h}{\tau}\cdot\lambda_1\cdot\beta_0-\gamma_1\cdot\left(\alpha_0-\frac{h^2}{\tau}\right)-0.5\frac{h}{\tau}\lambda_1\cdot F_0\cdot\left(\alpha_0-\frac{h^2}{\tau}\right)}{\left((1-\lambda_1)-0.5\cdot\frac{h}{\tau}\lambda_1\right)\cdot\left(\alpha_0-\frac{h^2}{\tau}\right)+\frac{h}{\tau}\lambda_1},\\
j=0...(N_x-1),\quad j=0...(N_y-1),\\
\vartheta_{i\,j+1}=\left(\dfrac{\alpha_{j+1}}{\frac{h^2}{\tau}-\alpha_{j+1}}\right)\cdot\left(\beta_{j+1}-\vartheta_{ij}\right)+\beta_{j+1},\quad i=1,2,3.
\end{array}\right\} \quad (14)$$

III. RESULTS AND DISCUSSION

To verify reliability, the results obtained for the developed algorithms have been compared with the results of trial functions.

The limits of applicability of this model and estimates of the values of $\|P_i - \vartheta_i\|_c$; $\|\vartheta_i - W_i\|_c$; $\|P_i - W_i\|_c$; are given in [14,16].

To analyze the results of the intermediate model and compare them with the calculation results obtained using the Hantusch model, the following initial data were taken as the basis for the calculations:

$P_i|_{t=0} = 100$ atm; $m_1 = m_2 = m_3 = 0.2$; $k_2 = 0.2$; $k_1 = k_3 = 0.05$; $h_0 = 0$; $h_1 = 1/12$; $h_2 = 11/12$; $h_3 = 1$; $\varphi_i = 1$; $\mu = 2$; $\Delta x = 0.01$; $\Delta y = 0.01$; $\Delta z = 0.02$; $\tau = \frac{Lx \cdot Lx}{86400}$; $Lx = 1000м$; $Ly = 1000м$; $Lz_2 = 100м$; $Lz_1 = Lz_3 = 50м$.

The values of the selections and the coordinates of the well locations are given in Table 1.

TABLE 1. WELL DATA.

№	X, m	У, m	$10^3 * \frac{m^3}{day}$	№	X, m	У, m	$10^3 * \frac{m^3}{day}$
1	200	200	10	9	600	200	10
2	200	400	10	10	600	400	10
3	200	600	10	11	600	600	10
4	200	800	10	12	600	800	10
5	400	200	10	13	800	200	10
6	400	400	10	14	800	400	10
7	400	600	10	15	800	600	10
8	400	800	10	16	800	800	10

The pressure distributions of the highly permeable formation and the values of the flow from the bridge are symmetrical relative to the average formation due to the symmetry of the location of the selected points (wells). Therefore, their values in a quarter of the field are given in Tables 2 and 3, respectively. For a complete presentation of their values, Fig. 2 presents them as a map of isolines at the time $T = 720$ days.

TABLE 2. VALUES OF HIGHLY PERMEABLE FORMATION.

Y \ X	X=0	X=100	X=200	X=300	X=400	X=500
Y=500	0.998	0.996	0.994	0.994	0.993	0.993
Y=150	0.998	0.996	0.992	0.993	0.991	0.993
Y=400	0.998	0.996	0.980	0.993	0.979	0.993
Y=350	0.998	0.996	0.992	0.993	0.991	0.993
Y=300	0.998	0.997	0.994	0.994	0.993	0.994
Y=250	0.998	0.997	0.993	0.994	0.992	0.993
Y=200	0.998	0.997	0.981	0.994	0.980	0.994
Y=150	0.999	0.997	0.994	0.995	0.993	0.995
Y=100	0.999	0.998	0.997	0.997	0.996	0.996
Y=50	0.999	0.999	0.998	0.998	0.997	0.997
Y=0	0.999	0.999	0.998	0.998	0.998	0.998

TABLE 3. FLOW VALUES.

Y \ X	X=0	X=100	X=200	X=300	X=400	X=500
Y=500	0.0000	0.0037	0.0072	0.0073	0.0084	0.0075
Y=150	0.0000	0.0039	0.0089	0.0077	0.0101	0.0079
Y=400	0.0000	0.0042	0.0242	0.0081	0.0254	0.0084
Y=350	0.0000	0.0039	0.0088	0.0076	0.0100	0.0078
Y=300	0.0000	0.0036	0.0069	0.0070	0.0081	0.0073
Y=250	0.0000	0.0036	0.0084	0.0071	0.0095	0.0073
Y=200	0.0000	0.0035	0.0232	0.0069	0.0242	0.0072
Y=150	0.0000	0.0027	0.0069	0.0055	0.0077	0.0057
Y=100	0.0000	0.0017	0.0035	0.0036	0.0042	0.0037
Y=50	0.0000	0.0010	0.0020	0.0023	0.0025	0.0024
Y=0	0.0000	0.0037	0.0072	0.0073	0.0084	0.0075

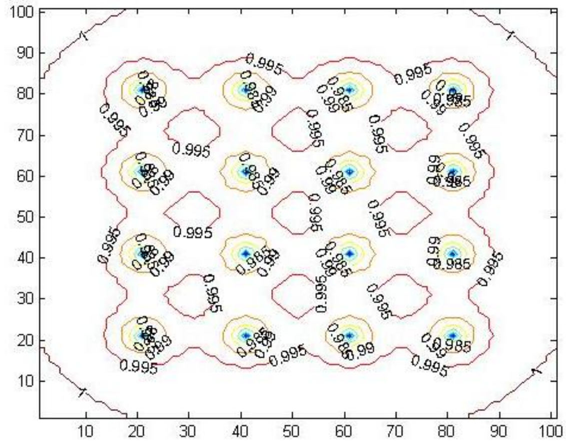

Fig.2. Isobar of a well-permeable reservoir

Now let’s see how the pressure value around the well changes over time. Considering the symmetry of the results, let's look at one of the neighboring points to the well at different values of the viscosity coefficient (table 4) for two models

TABLE 4. DIFFERENT VALUES OF THE VISCOSITY COEFFICIENT FOR TWO MODELS.

t day	Intermediate model		Hantush's model		Intermediate model		Hantush's model	
	$\mu = 2$				$\mu = 5$			
200	0.991	0.985	0.991	0.985	0.987	0.982	0.982	0.967
400	0.989	0.983	0.989	0.982	0.979	0.963	0.979	0.963
540	0.987	0.981	0.987	0.981	0.977	0.961	0.977	0.961
720	0.985	0.979	0.986	0.980	0.975	0.959	0.975	0.960

The following conclusions can be drawn from the given values. For both models in the considered example, the pressure values are almost the same. Therefore, with such data, it is advisable to use PM using a small amount of memory and less calculation time.

IV. CONCLUSIONS

This research comprehensively addresses the complexities of modeling and solving fluid flow dynamics in multilayer oil and gas fields, particularly under conditions involving hydrodynamic connections between layers. The study focuses on simplifying the intricate problem of pressure distribution and flow calculations in multilayer reservoirs using an intermediate model (IM) and comparing its efficacy with the more computationally intensive Hantush model.

In conclusion, this study demonstrates that the intermediate model is a robust and efficient alternative to traditional methods for simulating fluid flow in multilayer reservoirs. Its balance of accuracy, computational efficiency, and adaptability makes it an invaluable tool for optimizing the development and management of oil and gas fields. Future research can further enhance the model by incorporating more complex geological features and real-time data integration, enabling even more precise and dynamic field management.

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