Environment. Technology. Resources. Rezekne, Latvia
Proceedings of the 16th International Scientific and Practical Conference. 2025. Volume II, 181-187

Special Hyperbolic Type Approximation and Fourier Method for Solving 3-D Stationary Diffusion Problem

Ilmars Kangro
Rezekne Academy of Riga Technical University
Rezekne, Latvia
ilmars.kangro@rta.lv

Harijs Kalis
Institute of Mathematics and Computer Sciences
University of Latvia
Riga, Latvia
harijs.kalis@lu.lv

***Abstract*— In this paper, we explore the conservative averaging methods (CAM) used to solve three-dimensional boundary-value problems of second order. We investigate various interpretations of CAM to address these problems. We consider a special type of hyperbolic approximation - spline interpolation, that estimates middle integral values of piecewise smooth functions. By employing these splines, along with parabolic-type splines, we can reduce multidimensional mathematical physics problems in three dimensions to problems involving two dimensions for one coordinate. This approach also enables us to further simplify the two-dimensional problems into one-dimensional problems, where analytical solutions can be obtained. Additionally, we numerically solve the corresponding problem with homogeneous boundary conditions of the first kind using the Fourier series method, and we compare these numerical results with the analytical solutions.**

Keywords*— *Partial differential equation, conservative averaging method, 3-D boundary-value problem, 2-D and 1-D boundary-value problems.

1. Introduction

The task of sufficiently accurate numerical simulation of quickly solved 3-D problems for mathematical physics in multilayered media is important in known areas of the applied sciences. An example is the calculation of heavy metal concentration in the peat block—the layered peat block is modelled in [1], [2].

Reference [3] examines hyperbolic and parabolic heat conduction equations with a time-dependent heat source in a three-dimensional finite solid cube. It analyzes the thermal conductivity of a crystal and offers an analytical solution using the eigenfunction expansion method. The conservative averaging method (CAM) was developed as an approximate analytical and/or numerical method for solving a partial differential equation or its system with piece-wise constant (continuous) coefficients. To apply this method for all sub-domains of layered media, a special type of spline was constructed - a specially created spline allows the interpolation of the average values of piecewise integrals of smooth functions. The usage of this spline allows for diminishing the dimensions of the initial problem per one.

A. Buikis [4] had designed CAM with parabolic type splines whose applications were a mathematical simulation of the mass transfer process in multilayered underground systems - the mathematical foundation of this has been provided to him in the doctor of sciences thesis 1987 [5]. CAM as an approximate method for the solution of some direct and inverse heat transfer problems, where the solution is approximated with a polynomial, is given in [6], and a summary of the theories and applications of the problems to be solved has been analyzed in [7].

Paper [8] reviews the conservative averaging method's evolution over the last 50 to 100 years, focusing on layered media and introducing a new hyperbolic approximation for the method, along with a novel cubic spline representation.

Hyperbolic-polynomial splines, important in several applications, are a natural generalization of polynomial splines consisting of piecewise-defined functions with segments. Three-dimensional diffusion problems with discontinuous coefficients and one-dimensional Dirac sources are considered in [9]. We are analyzing (studying) a special spline with two different functions that interpolate middle integral values of piece-wise smooth functions. In [10], hyperbolic-type splines are used for solving heat and mass transfer 3-D problems in the porous multi-layered axial symmetry domain. Similarly, in [11] and [12], these splines are used for solving mass transfer problems in a

Online ISSN 2256-070X
https://doi.org/10.17770/etr2025vol2.8576

multi-layered domain. In [13], a special hyperbolic type approximation is used for the solving 3-D two-layer stationary diffusion problem, but in [14] for solving the combustion problem. In the book [15], some aspects for effective applied the conservative averaging methods for solving different problems of mathematical physics. We consider and propose a conservative averaging method, which is used for solving the stationary 3-D boundary-value problem (BVP) of second order with piece-wise coefficients.

The splines of the hyperbolic type we have created are discussed in this article. A specially created spline construction allows the interpolation of the average values of piecewise integrals of smooth functions.

With the help of the conservative averaging method, the original 3-D BVP was reduced first to a 2-D, and then to a 1-D BVP, and in this way, the approximate solution of the original 3-D BVP was obtained analytically (a functional relationship depending on the coordinate of any point in the definition area).

These splines contain one parameter in each of the three averaging directions, the choice of which determines the accuracy of the obtained solution.

2. Materials and Methods

The process of diffusion is considered in 3-D parallelepiped $\Omega = \{(x,y,z): 0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}$.
We will find the distribution of concentration $c = c(x,y,z)$ by solving the following 3-D boundary value problem for partial differential equation (PDE):

$$\begin{cases} \frac{\partial}{\partial x}\left(D_x \frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D_z \frac{\partial c}{\partial z}\right) + f_0 = 0, x \in (0, L_x), \\ y \in (0, L_y), z \in (0, L_z), \\ \frac{\partial c(0,y,z)}{\partial x} = \frac{\partial c(x,0,z)}{\partial y} = \frac{\partial c(x,y,0)}{\partial z} = 0, \\ D_x \frac{\partial c(L_x,y,z)}{\partial x} + \alpha_x (c(L_x,y,z) - c_{ax}) = 0, \\ D_y \frac{\partial c(x,L_y,z)}{\partial y} + \alpha_y (c(x,L_y,z) - c_{ay}) = 0, \\ D_z \frac{\partial c(x,y,L_z)}{\partial z} + \alpha_z (c(x,y,L_z) - c_{az}) = 0, \end{cases} \quad \text{(BVP1)}$$

where f_0 is the fixed source function, D_x, D_y, D_z are the constant coefficients, $\alpha_x, \alpha_y, \alpha_z$ are the constant mass transfer coefficients, c_{az}, c_{ay}, c_{ax} are the given concentration on the boundary.

A. The CAM with the hyperbolic type integral spline approximation in z-direction

Similarly, in [11] for solving (BVP1) we consider respect to the z-direction the CAM

$$c(x,y,z) = c_z(x,y) + m_z(x,y) f_{z1} + e_z(x,y) f_{z2} \quad (1)$$

$$f_{z1} = \frac{0.5 L_z \sinh(a_z(z - 0.5L_z))}{\sinh(0.5 a_z L_z)}, \quad (2)$$

$$f_{z2} = \frac{\cosh(a_z(z - 0.5L_z)) - A_{0z}}{8 \sinh^2(0.25 a_z L_z)}, \quad (3)$$

where $A_{0z} = \frac{\sinh(0.5 a_z L_z)}{0.5 a_z L_z}$, (4)

$c_z(x,y) = \frac{1}{L_z}\int_0^{L_z} c(x,y,z)dz$, (5) is the averaged value and a_z is fixed initial parameter (unknown).

We can see that the parameters a_z tend to zero then the limit is the integral parabolic spline from [4]. The unknown function $m_z(x,y), e_z(x,y)$ are determined from boundary conditions by $z = 0, z = L_z$: $d_z m_z - k_z e_z = 0$ (6), $m_z = p_z e_z$ (7),

$p_z = k_z / d_z$ (8), $d_z = 0.5 a_z L_z \coth(0.5 a_z L_z)$ (9),

$k_z = 0.25 a_z \coth(0.25 a_z L_z)$ (10),

$D_z(d_z m_z + k_z e_z) + \alpha_z(c_z + 0.5 m_z L_z + b_z e_z - c_{az}) = 0$ (11),

where $b_z = \frac{\cosh(0.5 a_z L_z) - A_{0z}}{8 \sinh^2(0.25 a_z L_z)}$. (12)

Therefore $e_z(x,y) = \frac{c_{az} - c_z(x,y)}{g_z}$, (13)

$g_z = b_z + 0.5 p_z L_z + 2 k_z D_z / \alpha_z$. (14)

The boundary value 2-D problem is in the following form:

$$\begin{cases} \frac{\partial}{\partial x}\left(D_x \frac{\partial c_z(x,y)}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_y \frac{\partial c_z(x,y)}{\partial y}\right) \\ + a_{0z}^2 (c_{az} - c_z(x,y)) + f_0 = 0, \\ \frac{\partial c_z(0,y)}{\partial x} = \frac{\partial c_z(x,0)}{\partial y} = 0, \\ D_x \frac{\partial c_z(L_x,y)}{\partial x} + \alpha_x (c_z(L_x,y) - c_{ax}) = 0, \\ D_y \frac{\partial c_z(x,L_y)}{\partial y} + \alpha_y (c_z(x,L_y) - c_{ay}) = 0, \end{cases} \quad \text{(BVP2)}$$

where $a_{0z}^2 = \frac{2 D_z k_z}{L_z g_z}$. (15)

B. The CAM with the hyperbolic type integral spline approximation in y-direction

For solving (BVP2) we are applying respect to y-direction of the CAM:

$c_z(x,y) = c_y(x) + m_y(x) f_{y1} + e_y(x) f_{y2}$ (16) - approximate solution with two functions

$$f_{y1} = \frac{0.5 L_y \sinh(a_y(y - 0.5L_y))}{\sinh(0.5 a_y L_y)}, \quad (17)$$

$$f_{y2} = \frac{\cosh(a_y(y - 0.5L_y)) - A_{0y}}{8 \sinh^2(0.25 a_y L_y)}, \quad (18)$$

where $A_{0y} = \frac{\sinh(0.5 a_y L_y)}{0.5 a_y L_y}$, (19)

$a_y = a_{0z}\sqrt{1/D_y}$. (20)

Similarly, as in the previous case - "the CAM in x - direction" we determine the unknown functions $m_y(x), e_y(x)$ from boundary conditions by $y = 0, y = L_y$ and we have

$e_y(x) = \frac{c_{ay}-c_y(x)}{g_y}$, (21)
$g_y = b_y + 0.5p_yL_y + 2k_yD_y/\alpha_y$, (22)
$m_y(x) = p_ye_y(x), p_y = k_y/d_y$, (23)
$d_y = 0.5a_yL_y\, coth(0.5a_yL_y)$, (24)
$k_y = 0.25a_y\, coth(0.25a_yL_y)$, (25)
$b_y = \frac{cosh(0.5a_yL_y)-A_{0y}}{8\, sinh^2(0.25a_yL_y)}$. (26)

Then we obtain the boundary value 1-D problem is in the following form

$$\begin{cases} \frac{\partial}{\partial x}\left(D_x\frac{\partial c_y(x)}{\partial x}\right) + a_{0y}^2\left(c_{ay} - c_y(x)\right) \\ +a_{0z}^2\left(c_{az} - c_y(x)\right) + f_0 = 0, \quad \text{(BVP3)} \\ \frac{\partial c_y(0)}{\partial x} = 0, D_x\frac{\partial c_y(L_x)}{\partial x} + \alpha_x\left(c_y(L_x) - c_{ax}\right) = 0, \end{cases}$$

where $a_{0y}^2 = \frac{2D_yk_y}{L_yg_y}$. (27)

C. The CAM with the hyperbolic type integral spline approximation in x-direction

For solving (BVP3) we are applying respect to x -direction the CAM:
$c_y(x) = c_x + m_xf_{x1} + e_xf_{x2}$ (28)- approximate solution with two functions
$f_{x1} = \frac{0.5L_x\, sinh(a_x(x-0.5L_x))}{sinh(0.5a_xL_x)}$, (29)
$f_{x2} = \frac{\cosh(a_x(x-0.5L_x))-A_{0x}}{8\sinh^2(0.25a_xL_x)}$, (30)
where $A_{0x} = \frac{sinh(0.5a_xL_x)}{0.5a_xL_x}$, (31)
$c_x = \frac{1}{L_x}\int_0^{L_x} c_y(x)dx$ (32) is the average integrals value, the parameter a_x we choose in the following form
$a_x = \sqrt{(a_{0z}^2 + a_{0y}^2)/D_x}$. (33)

Similarly, to the previously discussed cases of the CAM, we determine the unknown constants m_x, e_x from boundary conditions by $y = 0, y = L_y$ and we have
$e_x = \frac{c_{ax}-c_x}{g_x}$, (34)
$g_x = b_x + 0.5p_xL_x + 2k_xD_x/\alpha_x$, (35)
$m_x = p_xe_x$ (36), $p_x = k_x/d_x$ (37),
$d_x = 0.5a_xL_x \coth(0.5a_xL_x)$, (38)
$k_x = 0.25a_x \coth(0.25a_xL_x)$, (39)
$b_x = \frac{cosh(0.5a_xL_x)-A_{0x}}{8\, sinh^2(0.25a_xL_x)}$. (40)

Solving boundary value problem (BVP3) we obtain a linear algebraic equation

$$a_{0y}^2(c_{ay} - c_x) + a_{0z}^2(c_{az} - c_x) + a_{0x}^2(c_{ax} - c_x) + \\ +f_0 = 0 \quad (41)$$

or $c_x = \frac{f_0+a_{0y}^2c_{ay}+a_{0z}^2c_{az}+a_{0x}^2c_{ax}}{a_{0y}^2+a_{0z}^2+a_{0x}^2}$, (42)
$a_{0x}^2 = \frac{2D_xk_x}{L_xg_x}$. (43)

3. Results and discussion

In the previous chapter, it was obtained that the created splines contain one parameter in each of the three averaging directions, the choice of which determines the accuracy of the obtained solution.

In this chapter, a method was created to improve the accuracy of the obtained solution.

Accuracy can be improved by using different orientations of the averaging method, that is, by varying the averaging direction z, y and x in the dimensionality reduction process (3-D →2-D→1-D), thus obtaining the best parameter values that correspond to the optimal solution of the boundary value problem being solved.

The boundary value problem studied in the publication was solved, also using the Fourier method for homogeneous boundary conditions (BCs) of the first kind, and the numerical solutions obtained by both methods (the averaging method and the Fourier method) were compared.

A. The CAM with the hyperbolic type integral spline approximation in y -direction and z-direction

This chapter (*A.*) is studying the calculation of the parameters required for the conservative averaging method (CAM) using the direction change method in the process of implementing the CAM.

a) Calculation of $\boldsymbol{a_y}$ by applying averaging in y-direction

The approximate solution of the present stage of CAM is in the form:
$c(x,y,z) = c_y(x,z) + m_y(x,z)f_{y1} +$
$+e_y(x,z)f_{y2}$, (44)
where $c_y(x,z) = \frac{1}{L_y}\int_0^{L_y} c(x,y,z)dy$ (45) is the averaged value, functions f_{y1}, f_{y2} (46) are given in the section 2. *B.* Parameter a_y is in the form $a_y = a_{0z}\sqrt{1/D_y}$ (48), where
$a_{0z}^2 = \frac{2D_zk_z}{L_zg_z}$ (49).

The k_z and g_z functions depend on the previous value of a_z in the iteration process. Therefore, the a_z value for this parameter must be defined as the starting value for the iteration process.

b) Calculation of $\boldsymbol{a_z}$ by applying averaging in z-direction

The approximate solution of the present stage of CAM is in the form:

$c_y(x,z) = c_z(x) + m_z(x)f_{z1} + e_z(x)f_{z2}$, (50)

where $c_z(x) = \frac{1}{L_z}\int_0^{L_z} c_y(x,z)dz$ (51) is the averaged value, functions f_{z1}, f_{z2} (52) are given in section *A*. Parameter a_z is in the form $a_z = a_{0y}\sqrt{1/D_z}$ (53) where $a_{0y}^2 = \frac{2D_y k_y}{L_y g_y}$ (54).

The k_y and g_y functions depend on the a_y parameter value when averaging in y-direction (see *a)*).

This results in the new value of the a_z parameter in the iteration process.

c) Calculation of parameter $\boldsymbol{a_x}$

Parameter a_x is calculated using precalculated parameter values a_{0z}^2 and a_{0y}^2:

$$a_x = \sqrt{(a_{0z}^2 + a_{0y}^2)/D_x}. \quad (55)$$

The calculated values of a_z, a_y, a_x (see *a)*, *b)* and *c)*) correspond to one step in the iteration process.

The process continues by repeating *a)*, *b)* and *c)* using the parameter values calculated in the previous step.

The iteration process is completed when the maximum difference between the two step corresponding parameter values, taken from the absolute value, does not exceed the predefined accuracy.

Values of the iteration process results - calculated parameters a_z, a_y, a_x should be used for the application of CAM to solve the boundary value problem (BVP1) studied in the present publication (see Chapter 2).

B. The Fourier method for homogenous boundary conditions of first kind

We use the Fourier method to solve the boundary value problem (BVP) under question and to compare it with conservative averaging method (CAM) with the aim to assess the efficiency of the CAM.

For solving (BVP1) (see Chapter 2) we consider the extended domain

$\Omega_1 = \{(x_1,y_1,z_1): 0 \le x_1 \le 2L_x, 0 \le y_1 \le 2L_y, 0 \le z_1 \le 2L_z\}$

where $x_1 = x + L_x \in [L_x, 2L_x]$,

$y_1 = y + L_y \in [Ly, 2L_y], z_1 = z + L_z \in [L_z, 2L_z]$ for $x \in [-L_x, L_x], y \in [-L_y, L_y], z \in [-L_z, L_z]$.

We have homogenous boundary conditions (BCs) of first kind with

$\alpha_z = \alpha_y = \alpha_x = \infty$, $c_{az} = c_{ay} = c_{ax} = 0$. (56)

Then for the partial differential equation of (BVP1) (see Chapter 2) we have following BCs (without symmetrically BCs for $x = y = z = 0$):

$c(x_1, y_1, 2L_z) = c(x_1, 2L_y, z_1) = c(2L_x, y_1, z_1) = c(x_1, y_1, 0) = c(x_1, 0, z_1) = c(0, y_1, z_1) = 0$ (57)

and the solution of the (BVP1) in the series form is:

$c(x_1,y_1,z_1) = \sum_{i,j,k}^{\infty} A_{i,j,k} W_{i,j,k}(x_1,y_1,z_1)$, (58)

$f_0 = \sum_{i,j,k}^{\infty} B_{i,j,k} W_{i,j,k}$, (59)

where

$$B_{i,j,k} = f_0 \int_0^{2L_x}\int_0^{2L_y}\int_0^{2L_z} W_{i,j,k}(x_1,y_1,z_1)dx_1dy_1dz_1 = \frac{64 f_0\sqrt{L_xL_yL_z}}{\pi^3(2i-1)(2j-1)(2k-1)}, \quad (60)$$

$$W_{i,j,k}(x_1,y_1,z_1) = \sqrt{\frac{1}{L_xL_yL_z}}\sin\frac{i\pi x_1}{2L_x}\sin\frac{j\pi y_1}{2L_y}\sin\frac{k\pi z_1}{2L_z} \quad (61)$$

are the orthonormed eigenfunctions ($i,j,k = \overline{1,\infty}$) with the corresponding eigenvalues

$$\mu_{i,j,k} = \left(\frac{i\pi}{2L_x}\frac{j\pi}{2L_y}\frac{k\pi}{2L_z}\right)^2. \quad (62)$$

Therefore

$A_{i,j,k} = -C_0\sqrt{L_xL_yL_z}(-1)^{i+j+k}(a_{i,j,k})^{-1}$, (63)

$a_{i,j,k} = (2i-1)(2j-1)(2k-1)(a_i + a_j + a_k)$, (64)

$a_i = D_x((2i-1)/L_x)^2$, (65)

$a_j = D_y((2j-1)/L_y)^2$, (66)

$a_k = D_z((2k-1)/L_z)^2$, (67)

$C_0 = \frac{256 f_0}{\pi^5}$. (68)

We apply the Fourier series solution, performing averaging in different directions.

For averaging in xyz-direction by $z = L_z$ we have $c_{xyz} = \frac{1}{L_xL_yL_z}\int_{L_x}^{2L_x}\int_{L_y}^{2L_y}\int_{L_z}^{2L_z} c(x_1,y_1,z_1)dx_1dy_1dz_1$ (69)

and the discrete form of the solution is $c_{xyz} = \frac{8C_0}{\pi^3}\sum_{i,j,k}^{\infty}(b_{i,j,k})^{-1}$, $b_{i,j,k} = (2i-1)^2(2j-1)^2(2k-1)^2(a_i + a_j + a_k)$. (70)

For averaging in yz-direction by $x = L_x$ we have $c_{yz} = \frac{1}{L_yL_z}\int_{L_y}^{2L_y}\int_{L_z}^{2L_z} c(L_x,y_1,z_1)dy_1dz_1$ (71) and the discrete form of the solution is

$c_{yz} = -\frac{4C_0}{\pi^2}\sum_{i,j,k}^{\infty}(-1)^i(c_{i,j,k})^{-1}$, (71)

$c_{i,j,k} = (2i-1)(2j-1)^2(2k-1)^2(a_i + a_j + a_k)$. (72)

For averaging in z-direction by $x = L_x, y = L_y$ we have $c_z = \frac{1}{L_z}\int_{L_z}^{2L_z} c(L_x,L_y,z_1)dz_1$ (73) and the discrete form of the solution is

$c_z = \frac{2C_0}{\pi}\sum_{i,j,k}^{\infty}(-1)^{i+j}(d_{i,j,k})^{-1}$, (74)

$d_{i,j,k} = (2i-1)(2j-1)(2k-1)^2(a_i + a_j + a_k)$. (75)

The maximal value M_F is calculated as follows: $M_F = c(L_x,L_y,L_z) = -C_0\sum_{i,j,k}^{\infty}(-1)^{i+j+k}(a_{i,j,k})^{-1}$. (76)

Example 1.

We consider the (BVP1) (see Chapter 2) with parameters $f_0 = 1, L_z = 1, L_x = 1, L_y = 1, D_z = 10^{-3}, D_x = D_y = 3.10^{-4}$, $\alpha_z = \alpha_y = \alpha_x = \infty$, $c_{az} = c_{ay} = c_{ax} = 0$.

For initial value $a_z = 1$ we obtain with the iteration process using 5 iterations (*It*) the following parameter values $a_z = 1.1898, a_y = 3.3027, a_x = 3.9530$ for applying the CAM, difference (*konv*) (maximum difference between the two step corresponding parameter values, taken from the absolute value) is defined 10^{-6} (see Fig. 1, Table 1).

TABLE 1 ITERATION PROCESS RESULTS

Iteration steps	Parameters		
	a_z	a_y	a_x
1.	1.0000	3.2632	3.9158
2.	1.1856	3.3018	3.9521
3.	1.1897	3.3027	3.9530
4.	1.1898	3.3027	3.9530

Table 1 shows the first four iteration steps for coefficient values, written with four decimal places. The results of the fourth and fifth steps do not differ from the results of the third iteration step for the number of decimal places considered.

The maximal values (M) and averaged values (A) of the solution were compared, obtained by the Fourier method and by the spline method (hyperbolic, parabolic approximation):

1. Maximal values M ($M_F = 403.63$ for Fourier method with 10 series sum $M_H = 399.44$ for hyperbolic approximation, $M_P = 703.125$ for parabolic approximation);
2. Averaged values A for the following averaging directions:

a) xyz -direction ($Axyz_H = 159.39$ for hyperbolic approximation, $Axyz_F = 154.40$ for Fourier method, $A_P = 208.33$ for parabolic approximation);
b) yz -direction ($Ayz_H = 205.13$ for hyperbolic approximation, $Ayz_F = 203.37$ for Fourier method);
c) z -direction ($Az_H = 272.27$ for hyperbolic approximation, $Az_F = 271.96$ for Fourier method),
($a_x = a_y = a_z = 0.0001$ for parabolic spline).

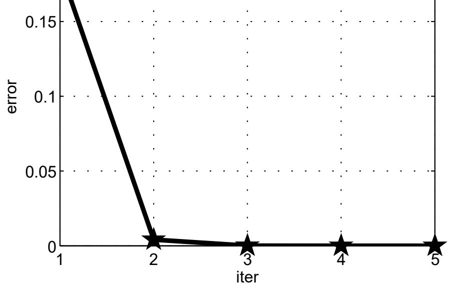

Fig. 1. Iteration process for a_z, error for *konv*=10^{-6}, $It = 5$.

In the Fig. 2 - Fig. 4 there are represented the hyperbolic and parabolic spline solutions $c(x, y, 0)$ and Fourier method's solutions $c(x_1, y_1, L_z)$.

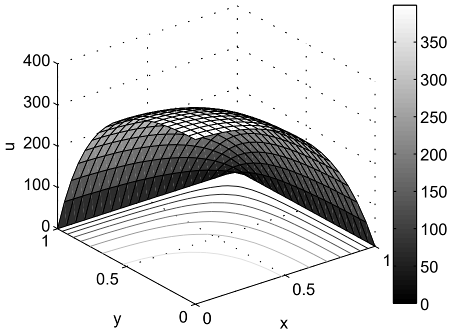

Fig. 2. Solution $u = c(x, y, 0)$ with hyperbolic approximation, $Max_H = 399.44$, $A_H = 159.39$.

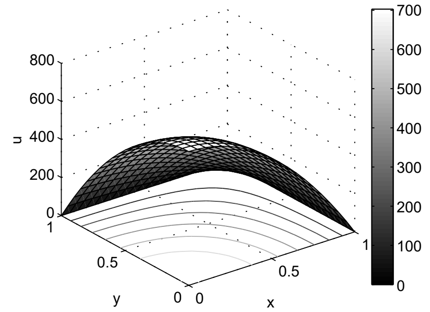

Fig. 3. Solution $u = c(x, y, 0)$ with parabolic approximation, $Max_P = 703.125, A_p = 208.33$.

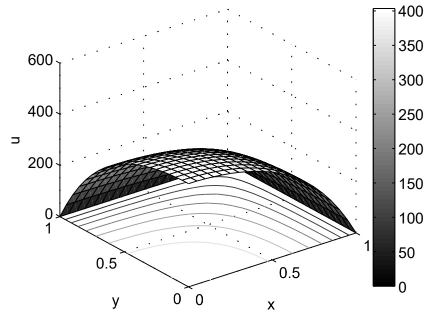

Fig. 4. Solution $u = c(x, y, 0)$ with Fourier approximation, $Max_F = 403.63, A_F = 154.40$.

To assess the accuracy of the results of the studying method, a relative error was used concerning another method (conditionally, more accurate).

Then the relative error of the results for the hyperbolic spline method, calculated relative to the result of the Fourier method, was approximately 1% for the value $M$and 3% for the value A (comparing the results of Fig. 2 and Fig. 4).

If $\alpha_x = \alpha_y = \alpha_z = 1$ (the case of the third kind of boundary conditions), then for hyperbolic splines we have obtained $M_H = 676.02, Axyz_H = 173.17$.
The maximal values (M) and averaged values (A) of the solution were compared also in the case of the third-kind of boundary conditions obtained by the spline method (hyperbolic, parabolic approximation) for BVP parameters $L_x = L_y = 1, L_z = 1$, $c_{ax} = c_{ay} = c_{az} = 0$, $\alpha_x = 1, \alpha_y = 20, \alpha_z = 300$, $D_x = 10^{-1}, D_y = 10^{-2}, D_z = 10^{-3}$ and for spline function's parameters $a_y = 0.6200, a_x = 0.2932, \alpha_z = 300$, thus obtained the result: $M_H = 800.78$, $Axyz_H = 204.15$, $M_P = 911.59$, $A_P = 213.87$.
Then the relative error of the results for the parabolic spline method, calculated relative to the result of the hyperbolic spline method, was 14% for the value M and 5% for the value A.

Example 2.
We consider the (BVP1) (see Chapter 2) with parameters $f_0 = 2, L_z = 1, L_x = 2, L_y = 2, D_z = 10^{-3}$, $D_x = 10^{-3}$, $D_x = 3.10^{-4}$, $D_y = 3.10^{-5}$, $\alpha_z = \alpha_y = \alpha_x = \infty$, $c_{az} = c_{ay} = c_{ax} = 0$. For the initial value $a_z = 1$ we obtain with the iteration process using 5 iterations the following parameter values $a_z = 0.3983, a_y = 10.0525, a_x = 3.2610$ for applying the CAM.
For comparing the maximal (M) and averaged (A) solution's values obtained by the Fourier method and by the spline method (hyperbolic approximation) the following results were calculated:
1. Maximal values M ($M_F = 992.16$ for Fourier method with 20 series sum, $M_F = 992.17$ for Fourier method with 40 series sum, $M_F = 1004.14$ for Fourier method with 10 series sum, $M_H = 984.08$ for hyperbolic approximation);
2. Averaged values A for the following averaging directions:
a) xyz -direction ($Axyz_H = 530.78$ for hyperbolic approximation, $Axyz_F = 528.05$ for Fourier method);
b) yz -direction ($Ayz_H = 625.06$ for hyperbolic approximation, $Ayz_F = 626.06$ for Fourier method);
c) z -direction ($Az_H = 657.78$ for hyperbolic approximation, $Az_F = 661.45$ for Fourier method).
3. $M_P = 2078.52$, A_P=615.86 for parabolic approximation.
There are represented the hyperbolic and Fourier methods solutions $c(x, y, 0)$ (see Fig. 5 and Fig. 6).

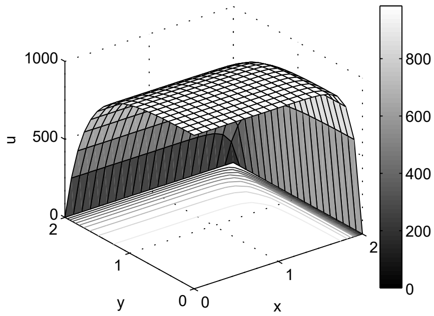

Fig. 5. Solution $u = c(x, y, 0)$ with hyperbolic approximation, $Max_H = 984.08, A_H = 530.78$.

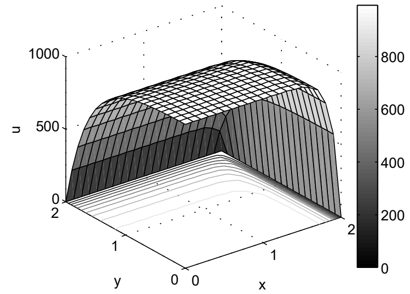

Fig. 6. Solution $u = c(x, y, 0)$ with Fourier approximation, $Max_F = 992.17, A_F = 523.05$.

In this case, the comparison results (Fig. 5 and Fig. 6) are as follows - the relative error of the results for the hyperbolic spline method, calculated concerning the result of the Fourier method, was approximately 1% for the value $M$and 2% for the value A .
Similar to the previously discussed cases, here too, the relative error obtained with the hyperbolic spline method concerning the result of the Fourier method does not exceed 3%.

4. CONCLUSIONS

1. Being based on the conservative averaging method (CAM), the special hyperbolic type splines with 2 fixed functions with unknown parameters for solving 3-D stationary diffusion boundary value problem (BVP) have been created.
2. It is characteristic that for the successful solution of the above-mentioned BVP, it is necessary to find the unknown parameters in the fixed functions, included in the splines. Therefore, in addition to the developed general scheme of the conservative averaging method (for example, averaging in the z ->y->x directions), it was necessary to create a special method for calculating the unknown parameters of spline functions. The developed method is based on varying the averaging directions, which allows a calculation of the necessary parameter values to obtain the optimal solution to the boundary value problem.
It should also be noted that varying the directions can be successfully applied to other types of splines, such as

exponential and parabolic, in solving boundary value problems.

3. To assess the accuracy of the results of the method under study, the relative error was used regarding another method (conditionally more accurate), for example, the relative error of the results of the hyperbolic spline method regarding the results of the Fourier method was calculated. In the case of first-kind boundary conditions (BCs), the relative error was calculated regarding the results of the Fourier method, but for the third kind BCs (comparing hyperbolic and parabolic spline) - regarding the results of the hyperbolic spline solutions.

In numerical experiments solving the studied boundary value problem, it was found that the relative error of the results for the hyperbolic spline method, calculated relative to the result of the Fourier method,

did not exceed approximately 1% for the maximal value M and 3% for the averaged value A - therefore, it is a completely acceptable result for engineering calculations.

4. It is also worth noting the excellent capabilities of the MATLAB software in solving the studied boundary value problem with the Fourier method in defining mass transfer coefficients and further using them as the operational algorithms. For the theoretical values $\alpha_z = \alpha_y = \alpha_x = \infty$ choosing $\alpha_z = \alpha_y = \alpha_x = 10^9$ in the corresponding practical solution, we obtain a solution to the boundary value problem that is completely correct for the theoretical framework and useful in practice.

5. The Solution of the studied boundary value problem obtained by the CAM in the analytical form allows for a better understanding and interpretation of the solution of the studied 3-D BVP alongside other possible methods of obtaining the solution only numerically.

REFERENCES

[1] E. Teirumnieka, E. Teirumnieks, I. Kangro, H. Kalis, A. Gedroics. The mathematical modeling of Ca and Fe distribution in peat layers. Proceed. Of the 8-th int. scienti_c practical conference "Environment. Technology. Resources." Rezekne higher education institution, June 20-22, 2 2011, 40-47.

[2] I. Kangro, H. Kalis, A. Gedroics, E. Teirumnieka, E. Teirumnieks. On mathematical modelling of metals distributions in peat layers. Mathematical Modelling and Analysis, 19:4 2014, 568-588.

[3] M. R. Talaee and V. Sarafrazi, Analytical solution for three-dimensional hyperbolic heat conduction equation with time-dependent and distributed heat source. Journal of Mechanics, DOI: 10.1017/jmech.2016.42, 2016, pp.1-11.

[4] A. Buikis, The analysis of schemes for the modelling same processes of filtration in the underground. Riga, Acta Universitatis Latviensis, 592, 1994, 25-32 (in Latvian).

[5] A. Buikis, Modelling filtration processes in layered porous media by the conservative averaging method. Dr. Thesis, Kazan, 1987 (in Rusian).

[6] A. Buikis, Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems. Advanced Computational Methods in Heat Transfer, WIT Transaction on Engineering Sciences, vol.53 2006, 311-320.

[7] A. Buikis, Conservative Averaging Method: Theory and Applications. Abstracts of MMA2015, May 26-29, Sigulda, Latvia, 2015, pp. 2.

[8] A. Buikis, The conservative averaging method development in the petroleum science. Journal of Scientific and Engineering Research, 2017, 4(4):176-197.

[9] E. Bejaoui and F. Ben Belgacem, Treatment of 3D diffusion problems with discontinuous coefficients and Dirac curvilinear sources, Appl. Numer. Math. 207, 2025, pp. 400–413.

[10] H. Kalis, A. Buikis, A. Aboltins, I. Kangro. Special Splines of Hyperbolic Type for the Solutions of Heat and Mass Transfer 3-D Problems in Porous Multi-Layered Axial Symmetry Domain. Mathematical Modelling and Analysis, 22:4 2017, 425-440.

[11] H. Kalis, A. Buikis and I. Kangro, Special splines of exponential type for the solutions of mass transfer problems in multi-layer domains. Mathematical Modelling and Analysis, 21:4 2016, 450-465.

[12] A. Buikis, H. Kalis and I. Kangro, Special hyperbolic type spline for mass transfer problems in multi-layer 3-D domains. Proc. of 3-rd int. conf. on applied, numerical and computational mathematics (ICANCM 15). Sliema, Malta, 2015, 25-34.

[13] I. Kangro, H. Kalis, E. Teirumnieka, E. Teirumnieks. Special Hyperbolic type approximation for solving of 3-D two-layer stationary diffusion problem. Proc. of the 12th int. scienti_c practical conference Environment. Technology. Resources., Rezekne Academy of Technologies, June 20-22, 2019, 95-100.

[14] A. Aboltins, I. Kangro and H. Kalis, Conservative averaging methods for solving some nonlinear heat transfer problems related to combustion. Proc. Of the 19th int. conf. Engineering for Rural Development, May 20-22, 2020, Jelgava, 2020, 1698-1705.

[15] H. Kalis and I. Kangro, Effective Finite Difference and Conservative Averaging Methods for Solving Problems of sMathematical Physics. Monography, Rēzekne: Rēzeknes Tehnoloģiju akadēmija, 2021.