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Decision Making Decision Making Based on Fuzzy Preference Relations

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***Abstract* – Many fuzzy versions of common multi-criteria decision making (MCDM) methods have been proposed to date. Among these methods, a special place is occupied by the method based on fuzzy preference relations (FPR). This method is fuzzy in nature and has no crisp analogue. The essence of the method is to evaluate preferences on pairs of alternatives. The source for evaluation is subjective judgments of expert specialists based on their knowledge and experience. The purpose of this article is to present in detail and clearly the theoretical foundations of this specific method in the context of multi-criteria decision making under conditions of highly uncertain initial information. Based on the initial assignments of the experts, using relevant computational procedures, the resulting preference scores for each of the alternatives are determined. These resulting scores are the basis for selecting the optimal alternative or ranking the alternatives by preference. The article presents two alternative versions of this method. The article presents three illustrative examples, whose purpose is to demonstrate the relevant computational procedures.**

Keywords* – *aggregation of fuzzy preference relations, expert assessment of fuzzy preference relation, fuzzy preference relation (FPR), fuzzy strict preference relation, fuzzy nondominance relation.

I. INTRODUCTION

In general, the selection of optimal decision in multi-criteria decision making (MCDM) problems can be performed based on two general principles:

1. Based on comparisons of criterion values for all alternatives.

2. Based on paired comparisons of alternatives by preference for each criterion, without using the criterion values.

As examples of methods of the first group, the PROMETHEE, ELECTRE, TOPSIS methods should be mentioned. The "classic" example of a method of the second group is the AHP method.

There are many extensions of the AHP method to a fuzzy environment. In all of these extensions, the preference scores are expressed in the form of some fuzzy numbers.

This paper uses a different approach to pairwise comparison of alternatives by preference. This approach is based on the direct construction and use of fuzzy preference relations (FPR) in MCDM problems. Unlike fuzzy versions of AHP, this approach uses the values of the membership function of the pairwise evaluation results to the relevant FPR to map the results of fuzzy preference evaluation on pairs of alternatives. Although these values are discrete, their semantics reflects the uncertainties in the experts' assignments.

II. MATHERIALS AND METHODS

As a working version, this article uses a conditional group problem of multi-criteria decision making by some criterion, including four alternatives. Two experts perform relevant assessments. A detailed description of the fuzzy preference relations used in the article is based on data presented in relevant literature sources.

A. Introduction to Fuzzy Relations

The concept of relations between elements of one set or between elements of different sets is a fundamental mathematical concept that is widely used in various scientific and practical fields.

In mathematics, a relation denotes some kind of connection between two objects in a set of such objects, which may or may not be supported [1]. Formally, a relation R on a set X can be considered as a set of ordered pairs (x_i, x_j) - members of R [2].

The relation R in these definitions is a binary relation. There are more complex n-ary relations, but this article considers only binary relations. A binary relation can also be defined on two sets X, Y of objects, subjects, etc. Then

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we are talking about ordered pairs of objects (x_i, y_j), $x_i \in X$, $y_j \in Y$.

The concept of fuzzy relations was first proposed in [3]. A detailed historical overview of the development and practical applications of fuzzy relations is given in [4]. The concept of a fuzzy relation is an extension of the standard concept of a relation. Fuzzy relations include uncertainties regarding the degree of effectiveness of these relations.

Definition 1 [5]. Fuzzy relation R is a mapping from Cartesian space $X \times Y$ to interval [0. 1], where the strength of mapping is expressed by the membership function of the relation

$$R \subseteq X, Y \text{ and } \mu_R = X, Y \rightarrow [0.1] \quad (1)$$

Let X and Y be two relevant sets. A fuzzy relation R between X and Y (or from X to Y) is a fuzzy set of the form

$$R = \{(x, y), \mu_R(x, y) / x \in X, y \in Y\} \quad (2)$$

If X=Y, R is a fuzzy relation on X. If X and Y are finite sets, $X = \{x_1, ..., x_m\}$, $Y = [y_1, ..., y_n]$, then the fuzzy relation can be represented in the form of a matrix (table)

$$R = \left[\mu_R(x_i, y_j)\right]_{m \times n}, \; i = 1, ..., m, \; j = 1, ..., n.$$

The value $\mu_R(x_i, y_j)$ is the value of the membership function, reflecting the strength of the relationship between the values of $x_i \in X$ and $y_j \in Y$. This value is written at the intersection of the i-th row and the j-th column of the matrix R.

Let two fuzzy relations R and S be defined on the Cartesian product of sets $X \times Y$. Let us define the most common operations on these relations.

1. Union $(R \cup S)$:

$$\mu_{R \cup S} = \max\left[\mu_R(x, y), \mu_S(x, y)\right] \quad (3)$$

2. Intersection $(R \cap S)$:

$$\mu_{R \cap S} = \min\left[\mu_R(x, y), \mu_S(x, y)\right] \quad (4)$$

3. Complement $(\overline{R})$:

$$\mu_{\overline{R}}(x, y) = 1 - \mu_R(x, y) \quad (5)$$

4. Containment $(R \subset S)$:

It can be stated that a fuzzy relation R is contained in a fuzzy relation S if and only if

$$R \subset S \equiv \mu_R(x, y) \leq \mu_S(x, y), \forall x, y \quad (6, a)$$

By analogy, we can state that a fuzzy relation S is contained in a fuzzy relation R if and only if

$$S \subset R \equiv \mu_S(x, y) \leq \mu_R(x, y), \forall x, y \quad (6, b)$$

Let us present some important properties of fuzzy relations on the set X.

1. Reflexivity:

$$(x_i, x_i) \in R, \; \forall x_i \in X \quad (7)$$

2. Symmetry:

$$(x_i, x_j) \in R \rightarrow (x_j, x_i) \in R, \; \forall x_i, x_j \in X \quad (8)$$

3. Transitivity:

$$(x_i, x_j) \in R, \text{ and } (x_j, x_k) \in R \rightarrow (x_i, x_k) \in R,$$
$$\forall x_i, x_j, x_k \in X \quad (9)$$

B. Using FPR in MCDM Problems

In this article, we use a special type of fuzzy relations – fuzzy preference relations (FPR) – to evaluate alternatives and select the optimal alternative in the MCDM problem. The idea of such an approach was first proposed by L. Zadeh in his work [6].

Fig. 1 shows the graph of the membership function FPR P.

The horizontal axis shows the gradations of the strength of preference on a pair of alternatives (a_i, a_j). These gradations can be expressed by the following linguistic terms:

0 - a_j absolutely preferred to a_i;

1 - a_j very strongly preferred to a_i;

2 - a_j strongly preferred to a_i;

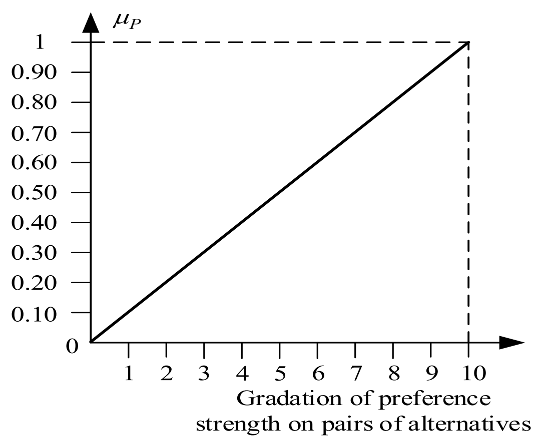

Fig. 1. Graph of the membership function μ_P FPR P.

3 - a_j moderately preferred to a_i;

4 - a_j slightly preferred to a_i;

5 - indifference between a_j and a_i;

6 - a_i slightly preferred to a_j;

7 - a_i moderately preferred to a_j;

8 - a_i strongly preferred to a_j;

9 - a_i very strongly preferred to a_j;

10 - a_i absolutely preferred to a_j.

The vertical axis shows the membership function's values corresponding to the subjective assignments of the expert's preference strength on pairs of alternatives. Using his knowledge and experience, the expert subjectively evaluates the preference strength on the relevant pair of alternatives. Using the membership function graph μ_P in Fig. 1, the value $\mu_P(a_i, a_j)$ can be determined. The set of values $\mu_P(a_i, a_j)$, $\forall a_i, a_j \in A$, form the initial data set in the MCDM problem.

The gradations 1–10 of the strength of preference in Fig. 1 are only guidelines for the expert to express his subjective judgments. In practice, he can assign any decimal number in the interval [0, 10] and determine the corresponding value of the membership function $\mu_P(a_i, a_j)$ in the interval [0, 1].

In general, FPR P can be either multiplicative $(p_{ij} * p_{ij} = 1)$ or additive $(p_{ij} + p_{ji} = 1)$. In this paper, we consider the additive FPR variant.

The main requirement for any set of results of pairwise comparisons of alternatives by preference is the consistency (compatibility) of these results. If an expert prefers alternative a_i to alternative a_j and he/she prefers alternative a_j to alternative a_k it is quite obvious that he should prefer alternative a_i to alternative a_k. In other words, the results of pairwise comparisons of alternatives should always be transitive.

Various types of transitivity on FPR are presented in [7]:

- weak transitivity:

$$p_{ij} \geq 0,5, \ \ p_{jk} \geq 0,5 \rightarrow p_{ik} \geq 0.5, \forall i,j,k \qquad (10)$$

- max-min transitivity:

$$p_{ik} \geq \min\left(p_{ij}, p_{jk}\right), \forall i,j,k \qquad (11)$$

- max- max transitivity:

$$p_{ik} \geq \max\left(p_{ij}, p_{jk}\right), \forall i,j,k \qquad (12)$$

- restricted max-min transitivity:

$$p_{ij} \geq 0,5, \ \ p_{jk} \geq 0,5 \rightarrow p_{ik} \geq \min\left(p_{ij}, p_{jk}\right), \forall i,j,k \ (13)$$

- restricted max-max transitivity:

$$p_{ij} \geq 0,5, \ \ p_{jk} \geq 0,5 \rightarrow p_{ik} \geq \max\left(p_{ij}, p_{jk}\right), \forall i,j,k \ (14)$$

- multiplicative transitivity:

$$\left(p_{ji} / p_{ij}\right) * \left(p_{kj} / p_{jk}\right) = p_{ki} / p_{ik} \forall i,j,k \ (15)$$

- additive transitivity:

$$\left(p_{ij} - 0,5\right) + \left(p_{jk} - 0.5\right) = \left(p_{ik} - 0.5\right), \forall i,j,k \ (16)$$

Equivalently, the additive transitivity property can be represented in the form [8].

$$p_{ij} + p_{jk} + p_{ik} = \frac{3}{2}, \forall i,j,k \qquad (17)$$

A detailed analysis of the above types of transitivity on FPR can be found in [9].

This paper uses a version of FPR with additive transitivity. In this version, a formal check of the compatibility of the results of pairwise comparisons can be performed according to the condition [9].

$$p_{i(i+1)} + p_{(i+1)(i+2)} + \ldots + p_{(j-1)j} + p_{ji} = \frac{j-1+1}{2}, \forall i < j. \ (18)$$

Testing under condition (18) can only confirm or disprove the compatibility of the relevant FPR. This naturally raises the question: is it possible in principle to construct an initial FPR that is compatible under condition (18). The following algorithm for solving this problem is proposed in [8].

Algorithm to construct a consistent reciprocal preference relation P' on $A = \{a_1, \ldots, a_n, n \geq 2\}$ from n-1 preference values $\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\}$ presents the following steps:

1. $B = \left\{p_{ij}, i < j \wedge p_{ij} \notin \left\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\right\}\right\}$,

   $p_{ij} = \frac{j-i+1}{2} - p_{i(i+1)} - p_{(i+1)(i+2)} - \ldots - p_{(j-1)j}$.

2. $a = \left|\left\{\min B \cup \left\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\right\}\right\}\right|$.

3. $P = \left\{p_{12}, p_{23}, \ldots, p_{(n-1)n}\right\} \cup B \cup$
   $\cup \left\{1 - p_{12}, 1 - p_{23}, \ldots, 1 - p_{(n-1)n}\right\} \cup \neg B.$

We will present the operation of this algorithm in an illustrative example later in this section.

In [9], an algorithm for constructing the multiplicative consistent fuzzy preference relation from an inconsistent fuzzy preference relation is proposed. This algorithm is not considered in this paper.

Let a group of experts perform pairwise comparisons of mutual preferences of alternatives. As a result, we have a set of individual FPRs. The problem of aggregating these individual FPRs into a generalized FPR, which we will call collective FPR, arises. Various methods have been proposed to solve this problem [10], [11], [12]. In this paper, we use the method presented in [12].

Let there be a group of m experts who performed pairwise comparisons of preferences of n alternatives. The results of their evaluations are reflected in the form of FPR $P_k = \left[p_{ij}^k \right]$, $p_{ij}^k + p_{ji}^k = 1$, $p_{ii}^k = 0.5$, i, j =1,..., n, k = 1,..., m. To aggregate individual FPR Pk into a collective FPR P^c, in [12] it is proposed to use the simple additive weighting (SAW) method.

$$P^c = \sum_{k=1}^{m} w_k P_k \qquad (19)$$

where $P^c = \left[p_{ij}^c \right]_{n \times n}$; w_k - weight of the k-th individual FPR.

To determine the weight vector $W = \left[w_1, w_2, ..., w_m \right]^T$, the following computational procedures must be performed.

The values of the differences are calculated as follows:

$$\delta_{ik} = \sum_{j=1, j \neq i}^{n} \left(p_{ij}^k - p_{ji}^k \right), \; i = 1,...,n, \; k = 1,...,m \qquad (20)$$

The calculation results are presented in the form of a matrix

$$\Delta = \begin{bmatrix} & P_1 & \dots & P_k & \dots & P_m \\ A_1 & \delta_{11} & \dots & \delta_{1k} & \dots & \delta_{1m} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_i & \delta_{i1} & \dots & \delta_{ik} & \dots & \delta_{im} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ A_n & \delta_{n1} & \dots & \delta_{nk} & \dots & \delta_{nm} \end{bmatrix}.$$

For each FPR P_k, the values of these two indices are calculated:

- absolute deviation:

$$AD_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \delta_{ik} - \delta_{jk} \right|, \; k = 1,...,m \qquad (21)$$

- standard deviation:

$$SD_k = \sqrt{\frac{1}{n} \delta_{ik}^2} \qquad (22)$$

FPRs with larger AD_k and SD_k values are more important. The weights determined based on AD_k values are called ADM weights in [12], and the weights determined based on SD_k values are called SDM weights. In this paper, we use SDM weights.

The values of SDM weights are calculated using the expression (23)

$$\tilde{w}_k = \frac{\left(\frac{1}{n} \sum_{i=1}^{n} \delta_{ik}^2 \right)^{\frac{1}{2(p-1)}}}{\left[\sum_{l=1}^{m} \left(\frac{1}{n} \sum_{i=1}^{n} \delta_{il}^2 \right)^{\frac{p}{2(p-1)}} \right]^{\frac{1}{p}}}, \; k = 1,...,m, \; p > 1 \qquad (23)$$

The normalized values of the SDM weights are calculated using the expression (24)

$$w_k = \frac{\left(\sum_{i=1}^{n} \delta_{ik}^2 \right)^{\frac{1}{2(p-1)}}}{\sum_{l=1}^{m} \left(\sum_{i=1}^{n} \delta_{il}^2 \right)^{\frac{1}{2(p-1)}}}, \; k = 1,...,m, \; p > 1 \qquad (24)$$

From (24) it follows that the normalized values of w_k depend on the values of the parameter p. In [12] it is recommended to use smaller values of p. In the present paper, the value p = 2 is used. With this value of p, expression (24) is transformed into the form

$$w_k = \frac{\sqrt{\sum_{i=1}^{n} \delta_{ik}^2}}{\sum_{l=1}^{m} \left(\sum_{i=1}^{n} \delta_{il}^2 \right)}, \; k = 1,...,m \qquad (25)$$

Note that for $p = \infty$ $w_k = \frac{1}{m}$, $k = 1,...,m$. In other words, the collective FPR is equal to the arithmetic mean of the individual FPRs.

Illustrative example 1. Let there be a set of alternatives $A = \{ a_1, a_2, a_3, a_4 \}$. The task of three experts is to assign their individual FPRs on this set of alternatives according to some criterion c_j. To solve this problem, the algorithm for constructing compatible FPRs presented above is used.

Expert E_1 performed the following initial assignments:

$$P_1^* = \begin{bmatrix} 0.50 & 0.53 & & \\ & 0.50 & 0.67 & \\ & & 0.50 & 0.73 \\ & & & 0.50 \end{bmatrix}.$$

The first task is to calculate the preference values in the lower left corner of the matrix. According to expression (18), we have

$p_{31} = 1.5 - p_{12} - p_{23} = 1.5 - 0.53 - 0.67 = 0.30$;

$p_{41} = 2 - p_{12} - p_{24} - p_{34} = 2 - 0.53 - 0.67 - 0.73 = 0.07$;

$p_{42} = 1.5 - p_{23} - p_{34} = 1.5 - 0.67 - 0.73 = 0.10$.

The remaining values are determined using the additivity property of FPR: $p_{ji} = 1 - p_{ij}$. We have

$$P_1 = \begin{bmatrix} 0.50 & 0.53 & 0.70 & 0.93 \\ 0.47 & 0.50 & 0.67 & 0.90 \\ 0.30 & 0.33 & 0.50 & 0.73 \\ 0.07 & 0.10 & 0.26 & 0.50 \end{bmatrix}.$$

Expert E_2 performed the following initial assignments:

$$P_2^* = \begin{bmatrix} 0.50 & 0.75 & & \\ & 0.50 & 0.80 & \\ & & 0.50 & 0.85 \\ & & & 0.50 \end{bmatrix}.$$

Let us carry out the calculations according to the above scheme.

$p_{31} = 1.5 - p_{12} - p_{23} = 1.5 - 0.75 - 0.80 = -0.05$;
$p_{41} = 2 - p_{12} - p_{24} - p_{34} = 2 - 0.75 - 0.80 - 0.85 = -0.40$;
$p_{42} = 1.5 - p_{23} - p_{34} = 1.5 - 0.80 - 0.85 = -0.15$.

$$P_2 = \begin{bmatrix} 0.50 & 0.75 & 1.05 & 1.40 \\ 0.25 & 0.50 & 0.80 & 1.15 \\ -0.05 & 0.20 & 0.50 & 0.85 \\ -0.40 & -0.15 & 0.15 & 0.50 \end{bmatrix}.$$

The initial estimates of expert E_2 are in the interval [-.40. 1.40], so these estimates must be transformed. According to procedure 2 of the algorithm, a = 0.40.

Let us calculate the transformed values according to procedure 4 of the algorithm.

$$p'_{12} = \frac{0.75 + 0.40}{1 + 2 * 0.40} = \frac{1.15}{1.80} = 0.64.$$

The remaining calculations are performed by analogy. Finally, we have

$$P'_2 = \begin{bmatrix} 0.50 & 0.64 & 0.80 & 1.00 \\ 0.36 & 0.50 & 0.67 & 0.80 \\ 0.20 & 0.33 & 0.50 & 0.69 \\ 0.00 & 0.20 & 0.31 & 0.50 \end{bmatrix}.$$

Using the initial assignments of expert E_3, the following FPR P_3 is formed

$$P_3 = \begin{bmatrix} 0.50 & 0.57 & 0.69 & 0.89 \\ 0.43 & 0.50 & 0.62 & 0.77 \\ 0.31 & 0.38 & 0.50 & 0.65 \\ 0.16 & 0.23 & 0.35 & 0.50 \end{bmatrix}.$$

We aggregate individual FPRs of experts. To do this, we first calculate the values δ_{i1} of the differences for the assignments of expert E_1 using (20).

$$\delta_{11} = (0.53 - 0.47) - (0.70 - 0.30) - (0.93 - 0.07) =$$
$$= 0.06 + 0.40 + 0.86 = 1.32;$$

$$\delta_{21} = (0.47 - 0.53) + (0.67 - 0.33) + (0.90 - 0.10) =$$
$$= -0.06 + 0.34 + 0.80 = 1.08.$$

By analogy, $\delta_{31} = -0.27$, $\delta_{41} = -2.13$.

The differences δ_{i2} and δ_{i3} for FPR P'_2, P_3 are calculated in a similar way. The resulting values of the differences are presented in the matrix Δ.

$$\Delta = \begin{bmatrix} & P_1 & P'_2 & P_3 \\ A_1 & 1.32 & 1.88 & 1.20 \\ A_2 & 1.08 & 0.66 & 0.64 \\ A_3 & -0.27 & -0.56 & -0.32 \\ A_4 & -2.13 & -1.98 & -1.52 \end{bmatrix}.$$

Using (25), we calculate the values of the FPR weights. First, we calculate the values of $\sqrt{\sum_{i=1}^{4} \delta_{ik}^2}$, $k = 1,2,3$.

$$\sqrt{\sum_{i=1}^{4} \delta_{i1}^2} = \sqrt{1.32^2 + 1.08^2 + (-0.27)^2 + (-2.13)^2} =$$
$$= \sqrt{1.74 + 1.17 + 0.08 + 4.54} = \sqrt{7.53} = 2.74.$$

By analogy,

$$\sqrt{\sum_{i=1}^{4} \delta_{i2}^2} = 2.86, \ \sqrt{\sum_{i=1}^{4} \delta_{i3}^2} = 2.06.$$

As a normalizing divisor in (25) we will take the sum of the values calculated above: 7.66. We have the following values of the FPR weights:

$$w_1 = \frac{2.74}{7.66} = 0.36, \ w_2 = \frac{2.86}{7.66} = 0.37, \ w_3 = \frac{2.06}{7.66} = 0.27.$$

Using (19), we determine the collective FPR Pc. To do this, we calculate the weighted average values for all p_{ijk}.

$$p_{12}^c = 0.36 * 0.53 + 0.37 * 0.64 + 0.27 * 0.57 = 0.58.$$

The remaining values p_{ij}^c are calculated by analogy. As a result, we have

$$P^c = \begin{bmatrix} 0.50 & 0.58 & 0.68 & 0.93 \\ 0.42 & 0.50 & 0.66 & 0.83 \\ 0.28 & 0.35 & 0.50 & 0.67 \\ 0.06 & 0.17 & 0.30 & 0.50 \end{bmatrix}.$$

C. Selection of Optimal Alternatives Based on FPR

Historically, the first approach to selecting optimal alternatives in MCDM problems where experts' preferences are represented in the form of FPR was the approach proposed in [13], [14]. We will present this approach based on the data from their work [15].

The fundamental concept in MCDM problems is the concept of the Pareto set, which is formed by alternatives that are not comparable in terms of criterion values. All these alternatives are called P-optimal.

By analogy with the concept of P-optimality, the concept of P-dominance can be introduced into consideration. If preferences in an MCDM problem are expressed in the form of FPR, these concepts can be expanded as follows.

Let P (or P^c) be an FPR reflecting fuzzy preferences (aggregated fuzzy preferences) on the initial set of alternatives according to some single criterion.

The following FPRs can be defined on the set of alternatives.

A fuzzy strict preference relation.

$$P^s = \frac{P}{P^{-1}} \quad (26)$$

with membership function values [13], [14]

$$\mu_{P^s}\left(a_i,a_j\right) = \max\left(\mu_P\left(a_i,a_j\right) - \mu_P\left(a_j,a_i\right),0\right) \quad (27)$$

A fuzzy non-dominance relation.

If $\left(a_i,a_j\right) \in P^s$, an alternative a_j is strictly dominated by an alternative a_i, $\mu_{P^s}\left(a_j,a_i\right)$, $\forall a_i \in A$, denotes the value of the membership function on pairs of alternatives $\left(a_j,a_i\right)$ in which the alternative a_i is strictly dominated by all alternatives a_j. The complement of this FPR, $\bar{P}^s = 1 - P^s\left(a_j,a_i\right)$, $\forall a_i \in A$ includes the set of alternatives which are not strictly dominated by others. This set is formed by those alternatives that belong to the non-dominance FPR P^{ND} with the values of the membership function defined by expression [13], [14].

$$\mu_{P^{ND}}\left(a_i\right) = \min\left(1-\mu_{P^s}\left(a_j,a_i\right)\right) = 1 - \max\left(\mu_{P^s}\left(a_j,a_i\right)\right) \quad (28)$$

The value $\mu_{P^{ND}}\left(a_i\right)$ reflects the degree to which the alternative a_i is not dominated by any of the alternatives of the set A.

The optimal alternative is the alternative a^{ND} that has the maximum value of the membership function $\mu_{P^{ND}}\left(a^{ND}\right)$ [13], [14].

$$a^{ND} = \left(a_i^{ND} \in A, \mu_{P^{ND}}\left(a_i^{ND}\right) = \max \mu_{P^{ND}}\left(a_i\right)\right) \quad (29)$$

If alternatives $a_i \in A$, $i = 1,...,m$, are evaluated by the values of a set of criteria $C = \left\{c_j \,/\, j = 1,...,n\right\}$, then the aggregated FPR P^G is defined as the intersection of the FPR Pj for each of the criteria [14]

$$P^G = \cap_{j=1}^{n} P_j \quad (30)$$

$$\mu_{P^G}\left(a_i,a_j\right) = \min_{1\le j\le n} p_j\left(a_i,a_j\right), \ \forall a_i,a_j \in A \quad (31)$$

In this case, the set of alternatives $\left\{a_i^{ND}\right\}$ is Pareto optimal set.

Illustrative example 2. It is necessary to determine the optimal alternative based on the resulting FPR P^c from illustrative example 1.

Let us define a fuzzy relation of strict preference on FPR P^c. To do this, using expression (27), we calculate the values of the membership function for pairs of alternatives $\left(a_i,a_j\right) \in A$.

$$\mu_{P^s}\left(a_1,a_2\right) = \max\left(\mu_{P^c}\left(a_1,a_2\right), \mu_{P^c}\left(a_2,a_1\right),0\right) =$$
$$= \max\left(0.58-0.42,0\right) = \max\left(0.16,0\right) = 0.16\,;$$
$$\mu_{P^s}\left(a_2,a_1\right) = \max\left(\mu_{P^c}\left(a_2,a_1\right) - \mu_{P^c}\left(a_1,a_2\right),0\right) =$$
$$= \max\left(0.42-0.58,0\right) = \max\left(-0.16,0\right) = 0\,.$$

The remaining values $\mu_{P^s}\left(\cdot,\cdot\right)$ are calculated by analogy. We have the following FPR P^s

$$P^s = \begin{bmatrix} 0.00 & 0.16 & 0.30 & 0.87 \\ 0.00 & 0.00 & 0.35 & 0.66 \\ 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}.$$

Using (28), we determine the values of the membership function of alternatives to the set of non-dominated alternatives.

$$\mu_{P^{ND}}\left(a_1\right) = 1 - \max\left(\mu_{P^s}\left(a_2,a_1\right), \mu_{P^s}\left(a_3,a_1\right), \mu_{P^s}\left(a_4,a_1\right)\right) =$$
$$= 1 - \max\left(0.00,0.00,0.00\right) = 1.$$

$\mu_{P^{ND}}\left(a_i\right)$, $i = 2,3,4$, are calculated by analogy.

$\mu_{P^{ND}}\left(a_2\right) = 0.84$, $\mu_{P^{ND}}\left(a_3\right) = 0.63$, $\mu_{P^{ND}}\left(a_4\right) = 0.13$.

The optimal is alternative a_1 that strictly dominates all other alternatives.

Another, simpler method for determining the optimal alternative based on FPR is proposed in [12]. The idea of this method is similar to the idea implemented in the TOPSIS method. Let a set of alternatives $A = \left\{a_i \,/\, i = 1,...,n\right\}$ be given and let FPR P be defined on this set. Then, for each alternative a_i, the input and output preference flows are calculated

$$\phi^+\left(a_i\right) = \sum_{j=1,j\ne i}^{n} p_{ij}\,,\ i = 1,..,n \quad (32)$$

$$\phi^-\left(a_i\right) = \sum_{i=1,i\ne j}^{n} p_{ji}\,,\ j = 1,..,n \quad (33)$$

The net flow of preferences for an alternative a_i is defined as

$$\phi(a_i) = \phi^+(a_i) - \phi^-(a_i),\ i = 1,..,n \quad (34)$$

If the alternatives are evaluated by a set of criteria $C = \{c_j / j = 1,...,m\}$, then m values $\phi_j(a_i)$ can be calculated. These values are aggregated in a suitable way, for example by weighted averaging

$$\phi^G(a_i) = \sum_{j=1}^{m} w_j \phi_j(a_i), \quad (35)$$

where w_j is the weight of the j-th criterion.

Illustrative example 3. It is necessary to determine the optimal alternative based on the above method, using the FPR P^c from illustrative example 1.

Let us perform the necessary calculations using (33), (34) and (35).

$$\phi^+(a_1) = 0.58 + 0.68 + 0.93 = 2.19;$$
$$\phi^-(a_1) = 0.42 + 0.28 + 0.06 = 0.76;$$
$$\phi(a_1) = 2.19 - 0.76 = 1.43.$$

The remaining calculations are performed by analogy. As a result, we have

$$\phi(a_1) = 1.43;\ \phi(a_2) = 0.81;\ \phi(a_1) = 1.43;$$
$$\phi(a_4) = -1.90.$$

The optimal alternative is a_1 with the highest value of the pure preference flow $\phi(a_1) = 1.43$.

III. Results and Discussion

This article presents a method for analyzing and selecting decisions based on fuzzy preference relations (FPR). This method is very similar in its ideology to fuzzy versions of the AHP method. In all these methods, fuzzy results of pairwise comparisons of alternatives by preference for each criterion are used as initial data. The difference is that in all fuzzy versions of the AHP method, the results of pairwise comparisons are expressed in the form of fuzzy numbers. Instead, in the method based on FPR, the values of the membership function to FPR, reflecting the degrees of preference on pairs of alternatives, are used as estimating values.

This paper presents two approaches to selecting optimal decisions based on FPR. The first approach is based on performing various specific operations on FPR. This approach requires complex and extensive calculations; whereas the second approach requires simpler calculations and is more visual for participants in the process of evaluating and selecting decisions.

The article provides simple examples to illustrate the computational procedures. An example of constructing a compatible FPR on a large set of elements (criteria) is given in [16]. A hybrid approach including AHP and FPR is presented in [17]. In [18], a hybrid approach including fuzzy VIKOR and FPR is discussed.

Examples of applying FPR to solving practical problems are given in [19], [20].

It should be noted that the decision-making method based on FPR has not found such wide practical use as fuzzy versions of other well-known MCDM methods. However, this method has great potential for its wider use in practical MCDM problems.

Conclusions

Various versions of multi-criteria decision-making methods have become very widespread in almost all areas of human activity in recent decades. Unfortunately, these methods have not yet become widespread in Latvia.

The use of fuzzy versions of MCDM methods expands the possibilities for the successful application of the corresponding methods in conditions of extremely uncertain initial information.

The main factors for increasing the efficiency of using MCDM methods for solving practical problems are: (1) the availability of high-level specialists who can professionally select and use suitable MCDM methods in real choice situations; (2) increasing the confidence of potential users in the efficiency of using formal MCDM methods for solving their specific problems.

One of the objectives of this article is to popularize one of the specific methods of fuzzy MCDM in order to expand the scope of use of this and other methods for solving practical problems in various areas of economic and social life in Latvia.

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